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On Gentle Perturbations, II

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ABSTRACT

This report is based on the observation that the gentleness condition defined with the aid of a Holder condition is ''local'' and that for perturbations of finite rank the Friedrichs' equation admits a ''formal solution''. This formal solution always can be interpreted as a densely defined, possible unbounded bilinear form. Then, roughly speaking, a solution which is gentle over some interval plays the same role in constructing the spectral transformation of the part of the operator over this interval, as a gentle solution did in constructing the spectral transformation of the entire operator.

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§1

INTRODUCTION

It is an interesting problem to ask for conditions which ensure that two operators acting on an abstract Hilbert space are unitarily equivalent. One expects that this is the case if their difference is small in some appropriate norm and thus one reformulates the question by asking which norms are appropriate? First this question was considered by von Neumann, [C1], who investigated the Hilbert-Schmidt norm and showed it to be inappropriate. Later, Friedrichs, [B2], gave a very simple example showing that an arbitrary small perturbation of rank one can produce pointeigenvalues. Hence there is no unitarily invariant crossnorm which is appropriate.

In spite of this large variety of inappropriate norms, he also introduced, [B2], an appropriate norm and called it a gentleness norm.

In this report we introduce the notion of an operator being gentle over an interval. For brevity, we shall refer to this interval as a gentleness interval and to the condition as local gentleness. This notion of local gentleness, which will be described in Section 2, is very general and at present we do not try to work with it. Instead, in Section 2, we also introduce a rather special space of locally gentle operators whose elements

are, in particular, operators of finite rank. Then we shall show t that for such locally gentle perturbations, the Friedrichs method can be applied to construct a spectral transformation for the part of the perturbed operator over a gentleness interval of the perturbation. Here and in the following we call the restriction of an operator to its eigenspace associated with a given interval, the part of the operator over the interval. In our construction we shall combine the gentleness considerations with the resolvent loop integral formula for the spectral projectors, [E2,c]. The important role of this formula in connection with gentleness considerations was emphasized recently by J. Schwartz, [B5], and according to a verbal communication of L.D. Faddeev, it will be emphasized in his forthcoming paper.

In Section 3, we shall introduce a sufficient condition in order that the Friedrichs' equation admits a locally gentle solution for locally gentle perturbations of finite rank. We shall see that, in general, this solution will be a densely defined bilinear form. In Section 4, we shall consider locally gentle perturbations of rank one and shall show the following: Under the conditions of Section 3, a spectral transformation of the part of the disturbed operator over a gentleness interval can be constructed with the aid of the solution to the Friedrichs' equation. We shall establish this by explicitly evaluating the

resolvent loop integral. In Section 5, we shall show that the results of Section 4 concerning locally gentle perturbations of rank one are typical for perturbations of finite rank. This will be the statement of the basic Theorem 5.1, of which we shall give several ramifications. In Section 6, we shall combine the theorems of Section 5 with Friedrichs' original theorem on ''small perturbations''. This will give theorems on perturbations which can be written as the sum of a gentle perturbation and a locally gentle perturbation of finite rank. This will yield a generalization of the main theorem of the previous paper, [B6]. In Section 7, we shall illustrate how our abstract theorems apply to certain differential operators.

Finally let us mention that the gentleness norm can be replaced by a unitarily invariant cross norm in order to draw the following slightly weaker conclusion: <u>The absolutely continuous parts of the perturbed and unperturbed operators are unitarily equivalent.</u> For, T. Kato, [A2,A3] has shown, using a lemma of M. Rosenblum [A1], that this is the case for trace class perturbations. Then M.G. Krein and M.S. Birman, [A6], showed that this also holds for relative trace class perturbations. In Appendix II, we shall illustrate the connection between the Kato theorem and the gentleness considerations by establishing a special case of his theorem using gentleness arguments.

2.1

§ 2

The space of locally gentle bilinear forms.

Originally the notion of gentleness was defined for operators by first defining it for kernels and then saying that an operator is gentle if it admits a gentle kernel. We shall proceed analogously in defining the notion of a locally gentle bilinear form. First, however, let us formulate some definitions concerning forms on an abstract Hilbert space $\mathcal{H}$.

Let F be a densely defined possibly unbounded form on $\mathcal{D}_1 \times \mathcal{D}_2$, and let B be a bounded operator. Then we define BF, the product of the operator B and the form F to be the form determined by $BF[f,g] = F[B^*f,g]$, where B^* is the adjoint of B. This form is defined for those vectors f for which B^*f is in $\mathcal{D}_1$. Since the range of a bounded operator need not be closed, and the intersection of two dense sets may be empty, it may happen that the form BF is defined for $f = 0$ only. Similarly we define the form FB by setting $FB[f,g] = F[f,Bg]$. Finally recall that F^*, the adjoint of the form F, is defined by $F^*[f,g] = \overline{F[g,f]}$.

Next we turn to the description of the space of locally gentle bilinear forms:

1. Let $\mathcal{H}$ be a seperable Hilbert space and A on $\mathcal{D}$ in $\mathcal{H}$ a strictly selfadjoint operator with continuous spectrum. Let $[\delta_1,\delta_2]$ be a given interval and E_δ the spectral projector of A over δ.

2.2

2. Let $\mathfrak{G}_{aux}$ be a collection of densely defined possibly unbounded forms on $\mathfrak{H}$ such that: a) The domain of each G in $\mathfrak{G}_{aux}$ is of the form $\mathfrak{D} \times \mathfrak{H}$, where $\mathfrak{D}$ is dense and possibly dependent on G. b) The finite sum of forms in $\mathfrak{G}_{aux}$ is also densely defined.

3. Suppose that for arbitrary E_δ and G in $\mathfrak{G}_{aux}$ the form $E_\delta G$ is in $\mathfrak{G}_{aux}$. Moreover there is a transformation Γ which assigns to a form G in $\mathfrak{G}_{aux}$ another densely defined form ΓG, possibly not in $\mathfrak{G}_{aux}$.

4. Let $\mathfrak{G}$ be a subset of $\mathfrak{G}_{aux}$ such that for every G in $\mathfrak{G}$ the form $E_\delta \Gamma G = \Gamma (E_\delta G)$ is bounded. Moreover $\Gamma G^* E_\delta = - \Gamma (G^* E_\delta)$.

We say that such a <u>quadruplet</u> $A, \mathfrak{G}, \mathfrak{G}_{aux}$, <u>and</u> Γ <u>defines a space of gentle bilinear forms over the interval</u> $[\delta_1, \delta_2]$ if the following three propositions hold:

P_1: <u>To every</u> G <u>in</u> $\mathfrak{G}_{aux}$, <u>there is a dense set such that on this set</u>

$$A \Gamma G - \Gamma G A = G .$$

P_2: <u>For every G in $\mathfrak{G}$ the forms</u> $E_\delta \Gamma G G^* E_\delta$ <u>and</u> $(E_\delta G) \Gamma (G^* E_\delta)$ <u>are in</u> $\mathfrak{G}_{aux}$. Note that $E_\delta \Gamma G G^* E_\delta$ is the product of the operator $E_\delta \Gamma G$, which according to No.4 is bounded, and of the form $G^* E_\delta$.

P_3: <u>For every G in</u> $\mathfrak{G}$

$$\Gamma (E_\delta G) \Gamma (G^* E_\delta) = \Gamma (E_\delta G \Gamma (G^* E_\delta) + \Gamma (E_\delta G) G^* E_\delta).$$

Note that according to proposition P_2 the form on the right side is well defined. Also note that the product of the two forms on the right side is well defined, for according to No.4 the form $\Gamma(E_5 G)$ can be identified with a bounded operator.

This notion of local gentleness is too general; therefore we introduce simplifying assumptions on A. We shall assume that the spectrum of A is of uniform multiplicity, moreover the spectral measure is absolutely continuous. Then according to the spectral theorem for strictly selfadjoint operators this is equivalent to:

Condition 2.1

The operator A is unitarily equivalent to M, the multiplication operator on $\mathcal{L}_2(\lambda, \mathfrak{H})$, where λ is the restriction of the Lebesgue measure to some closed set.

For brevity from now on we assume that A=M, and we turn to the description of a special space of locally gentle bilinear forms. For this purpose we need the notion of the dyadic product of two, possibly unbounded linear forms. Let $\mathfrak{D}_1$ be a dense set in $\mathfrak{H}$ and let $\mathfrak{D}_1^\dagger$ be the set of possibly unbounded forms on $\mathfrak{D}_1$. Similarly define the sets $\mathfrak{D}_2$ and $\mathfrak{D}_2^\dagger$. The dyadic product of the pair $f_1 \in \mathfrak{D}_1^\dagger$ and $f_2 \in \mathfrak{D}_2^\dagger$ is the form on $\mathfrak{D}_1 \times \mathfrak{D}_2$ defined by

$$(f_1 >< f_2)(g_1, g_2) = < f_1, \phi_1 > < f_2, \phi_2 > \quad .$$

Here $< f_1, g_1 >$ is the value of the form f_1 applied to g_1. In case the forms f_1 and f_2 are bounded they can be identified with vectors f_1 and f_2 in $\mathfrak{H}$, and similarly the bilinear form $f_1 > < f_2$ can be identified with an operator on $\widetilde{\mathfrak{H}}$. Then this operator is the dyadic product of the vectors f_1 and f_2 and we write

$$f_1 > < f_2 = f_1 \otimes f_2 .$$

The dyads entering the description of our space of locally gentle bilinear forms will be more special, for, they will be defined on an $\mathcal{L}_2$-space, instead of an abstract Hilbert space, and so we can require that the 1-forms entering them can be identified with measurable functions. More specifically let f be such a 1-form. Then we require that the domain of f is of the form $\bigcup \mathcal{L}_2(S_n)$, and that there is a measurable function $f(x)$ such that $|f(x)| \leq n$ on S_n moreover

$$< f,g > = \int f(x),g(x)\, d\lambda(x) \quad \text{for} \quad g \in \mathcal{L}_2(S_n) .$$

For brevity we call such a 1-form a measurable 1-form, and the dyadic product of two such 1-forms, a measurable 2-form of rank one. Measurable forms have the important feature that they are densely defined and their sum is also densely defined.

Once a form can be identified with a measurable function we can assign the properties of the measurable function to

the corresponding form. For example let f be a λ-measurable function defined with the possible exception of a set of λ-measure zero. We shall say that f is Hölder continuous if after setting $f=0$ on the exceptional set, the extended function is Hölder continuous on $(-\infty,+\infty)$. Using this notion of Hölder continuity we shall also say that the corresponding form, or functional is Holder continuous.

<u>The space $\mathfrak{S}(\alpha/[\delta_1,\delta_2], \lambda, \tilde{\Pi})$</u>

Let $[\delta_1,\delta_2]$ be a given bounded interval. Then we first define the space $\mathfrak{S}_{aux}$ to be the set of those measurable 2-forms of finite rank which can be written as

$$G = \sum h_i >< g_i \qquad g_i \in \mathcal{L}_2(\lambda, \tilde{\Pi}) .$$

Next define $\mathfrak{S}(\alpha/[\delta_1,\delta_2], \lambda, \tilde{\Pi})$ to be the subset of $\mathfrak{S}_{aux}$ consisting of those forms to which there is an interval $[\tilde{\delta}_1,\tilde{\delta}_2] \supset [\delta_1,\delta_2]$, such that the form is α-Hölder continuous on $[\tilde{\delta}_1,\tilde{\delta}_2]$; that is the functions $\{h_i\}$ and $\{g_i\}$ are α-Hölder continuous. Note that in view of our extension convention, in the special case $S_\lambda = [\delta_1,\delta_2]$, this condition requires in particular that the functions vanish at the end points. To complete the gentleness structure set

$$\Gamma_\varepsilon G(x,y) = \frac{G(x,y)}{x - y + i\varepsilon} \tag{2.1}$$

and define

$$< f, \Gamma Gg > = \lim_{\varepsilon \to +0} < f, \Gamma_\varepsilon Gg > , \text{ for } f,g \in \mathcal{L}_2(S_n).$$

Finally in case $[\delta_1, \delta_2]$ is an unbounded interval define the form G to be gentle over $[\delta_1, \delta_2]$ if it is gentle over any bounded subinterval. In Appendix I, we shall show that the quadruplet M, Γ, $\mathfrak{G}_{aux}$, and $\mathfrak{G}(\alpha/[\delta_1, \delta_2], \lambda, \mathfrak{H})$ defines a space of gentle forms over the interval $[\delta_1, \delta_2]$; that is we shall establish the validity of the propositions stated in the beginning of this section.

Note that our space of locally gentle forms overlaps with, but does not contain the corresponding space of gentle operators. For, we have required explicitly that our locally gentle forms are of finite rank, while a gentle operator need not be of finite rank. We made this simplifying assumption in view of the fact that for such forms the condition of local gentleness and the theorems concerning it, can be established in a particularly simple manner. Let us also note that the definition of the form ΓG is not restricted to forms of finite rank. Nevertheless if G is of finite rank the form ΓG can be described with the aid of the Hilbert transformation, and we can obtain properties of ΓG from the corresponding properties of the Hilbert transformation. More specifically we claim that if $G = g_1 >< g_2$ then

$$(2.2) \qquad < f_1, \Gamma G\, f_2 > = < f_1, g_1 H_+ g_2 f_2 >$$

where

$$H_+ \, g_2 f_2(x) = \int \frac{(g_2(y), f_2(y))}{y-x} \, d\lambda(y) + i\pi(f_2(x), g_2(x)) \ .$$

Finally we shall need the following generalization of the notion of a locally gentle function:

<u>The space of locally polite functions</u>,

$\mathcal{P}(\alpha/[\delta_1,\delta_2],\gamma,\lambda,\Gamma)$.

Let the function p be measurable with respect to the measure λ and set as before $p=0$ on the exceptional set. Suppose that p is such that there is a point x_0 in $[\delta_1,\delta_2]$ and a number ζ, satisfying the conditions

a) p is α-Hölder continuous in the intervals $[\delta_1,x_0-\zeta]$, $[x_0+\zeta,\delta_2]$.

b) for every $\tilde{\gamma}>|\gamma|$ the function $(x-x_0)^{\tilde{\gamma}}p(x)$ is α-Hölder continuous in $[\delta_1,\delta_2]$.

Then we define the space $\mathcal{P}(\alpha/[\delta_1,\delta_2],\gamma,\lambda,\Gamma)$ to be the set of finite linear combinations of such functions.

3.1

Solution of the Friedrichs' equation for locally gentle perturbations of finite rank.

According to the sonsiderations of Friedrichs, [B2,B7], one can construct a spectral transformation of the operator M+K, with the aid of the solution of the equation

$$(3.1) \qquad (1+\Gamma Q)K = Q \quad .$$

Originally one was seeking a solution Q to this equation in the class of gentle operators, a class for which ΓQ was a bounded operator. As mentioned earlier, this report is centered around the observation that it is useful to know whether this equation has a solution in the class of locally gentle forms and we shall take up this question in the present section. We start with a limiting case:

a) Case of arbitrary selfadjoint perturbations of finite rank.

We maintain that for perturbations of finite rank equation 3.1 always admits a solution Q in $\mathfrak{E}_{aux}$. More specifically we maintain that there exists a form Q in $_{aux}$ such that the product of the form ΓQ and the operator K, ΓQK is densely defined, moreover equation 3.1 holds on a dense set.

First let us assume that this is the case and let us

determine Q. It is well known that an arbitrary selfadjoint operator of finite rank, K, can be written as;

(3.2) $$K = \sum_{i=1}^{n} \lambda_i \, k_i \otimes k_i \ ,$$

where the $\{k_i\}$ are the normalized eigenvectors and the $\{\lambda_i\}$ are the corresponding eigenvalues. Insertion of this fact in (3.1) yields that,

(3.3) $$Q = \sum h_i >< \lambda_i k_i \quad .$$

Inserting this relation in (3.1) again and using that the functions $\{k_i\}$ are linearly independent, we obtain a system of equations;

(3.4) $$1 + \Gamma\Big(\sum_j h_j >< \lambda_j k_j\Big)k_i = h_i$$
$$i = 1,2,\ldots n \quad .$$

By assumption Q is in $\mathfrak{G}_{aux}$, hence each of the forms $\{h_i\}$ can be identified with a measurable function which we denote by $\{h_i\}$ again. Then according to the previous section, the form $\Gamma(h_j >< k_j)k_i$ can be identified with a measurable function, moreover this measurable function can be expressed in terms of the augmented Hilbert transformation;

(3.5) $$\Gamma(h_j >< k_j)k_i(x) = h_j(x)H_+(k_j(\cdot),k_i(\cdot))(x) \quad .$$

Finally, inserting this relation in (3.4) we obtain the

3.3

following system of linear equations for the unknown functions $\{h_i\}$;

(3.6) $\quad A(x)h(x) = k(x)$

with

$$A_{ij}(x) = \delta_{ij} + \lambda_j\, H_-(k_i(\cdot),k_j(\cdot))(x)$$

(3.7) $$= \delta_{ij} + \lambda_j \left(\int \frac{(k_i(y),k_j(y))\ldots}{y-x}\, d\lambda(y) - i\pi(k_i(x),k_j(x)\right)$$

where δ_{ij} denotes the Kronecker symbol. Hence in (3.3) and (3.6) we have found an equation for the solution Q.

Note that the vectors $k(x)$: $\{k_i(x)\}$, $h(x)$: $\{h_j(x)\}$ are elements of the n-orthogonal sum of $\tilde{\mathfrak{H}}$ with itself, which we denote by $\tilde{\mathfrak{H}}^{(n)}$. Equation (3.6) is an equation on $\tilde{\mathfrak{H}}^{(n)}$, $A(x)$ maps $\tilde{\mathfrak{H}}^{(n)}$ into $\tilde{\mathfrak{H}}^{(n)}$ and the associated matrix is a numerical valued matrix which we denote by $A(x)$ again. Also note that this matrix is defined almost everywhere only as a Cauchy principle value.

Next we claim that the form Q defined by equations (3.3) and (3.6) is in $\mathfrak{D}_{aux}$, which amounts to the statement that the functions $\{h_i\}$ entering it are measurable. In order to establish this fact we first note that the matrix $A(x)$ is measurable. For, according to the $\mathfrak{L}_1$-version of the Plemelj-

Privalov theorem, [E3], this matrix is the a.e. boundary value of the matrix

$$A_{ij}(z) = \delta_{ij} + \lambda_j \int \frac{(k_i(y), k_j(y))}{y-z} d\lambda(y) .$$

Hence almost everywhere it is the pointwise limit of a family of measurable matrices and so it is measurable. Next we note that this matrix admits a.e. an inverse which is measurable in view of the fact that A(x) is measurable. At this point we make essential use of the fact that A(z) can be identified with the matrix of the perturbation, (M, M+K), and according to S.T. Kuroda, the boundary value of the perturbation matrix is almost everywhere invertible, [A8]. Therefore, the functions $\{h_i\}$ defined by equation (3.6) are measurable and hence the form Q defined by (3.3) is in $\mathfrak{G}_{aux}$.

The arguments leading to equations (3.3) and (3.6) show that in order that a form Q in $\mathfrak{G}_{aux}$ should satisfy equation (3.1) it is necessary that (3.3) and (3.6) should hold. The same argument also shows that for a form Q in $\mathfrak{G}_{aux}$ this condition is also sufficient. Therefore, from the statement of the previous paragraph we conclude that for an <u>arbitrary selfadjoint perturbation of finite rank the Freidrichs' equation,</u>

(3.11), admits a solution in $\mathfrak{G}_{aux}$. Most likely this is all that one can say about the solution in general; therefore, we introduce additional assumptions in the following.

b) Case of locally gentle perturbations of finite rank.

First let us assume that the previous operator K is gentle over some bounded interval $[\delta_1,\delta_2]$. Then we ask the question, Does this ensure that the solution h(x) to (3.6) is gentle over $[\delta_1,\delta_2]$? From the local gentleness of the eigenfunctions we infer in view of (3.7) the local Holder continuity of the matrix A(x). Now this matrix has two types of exceptional points. Those at which the integrals defing the the matrix elements fail to exist in the sense of Cauchy's principle value and those at which the matrix A(x) is not invertible. For brevity let us call them exceptional points of the first and second type. Now the gentleness of K over $[\delta_1,\delta_2]$ ensures that the matrix A(x) has no exceptional points of the first type in $[\delta_1,\delta_2]$. Nevertheless we maintain that exceptional points of the second type may exist in the interval $[\delta_1,\delta_2]$. For, in the previous paper [B6] we gave an example of a gentle perturbation of rank one for which the matrix A(x) was not invertible, moreover the Friedrichs' equation had no gentle

solution. We also introduced a condition there, which excluded such examples. Now we observe that this condition is local, for, in a slightly generalised form, it reads as follows:

Condition $[\delta_1, \delta_2]$

The set of pointeigenvalues of M+K on $\mathfrak{H}(\alpha, \alpha-1, \lambda, \mathfrak{N})$ is disjoint from the closed interval $[\delta_1, \delta_2]$.

Next we shall establish the fact that under this condition the previously constructed solution to equation (3.1) is gentle over the interval $[\delta_1, \delta_2]$ which we assume to be bounded. More specifically, we shall establish the following theorem:

Theorem 3.1

Let K be a self-adjoint operator of finite rank on $\mathfrak{L}_2(\lambda, \widetilde{\mathfrak{N}})$ which is gentle over the interval $[\delta_1, \delta_2]$. Suppose that the operator M+K satisfies Condition $[\delta_1, \delta_2]$. Then for the operator K the Friedrichs' equation, (3.1), admits a gentle solution in the interval $[\delta_1, \delta_2]$.

This theorem is a slight generalization of a previous theorem, [B6], inasmuch as it is local and there is no restriction on the Holder exponent c entering the definition of gentleness. The prrof is quite analogous to the previous proof, nevertheless for the sake of completeness let us carry out the details.

Since by assumption $[\delta_1, \delta_2]$ is a bounded closed interval

the quotient of two functions which are gentle in this interval is gentle provided that the denominator does not vanish. Hence the functions $\{h_i\}$ entering equation (3.6) will be gentle provided that the matrix (3.7) is invertible. Therefore, in order to establish the theorem, it suffices to establish that the matrix A(x) is locally invertible. Now it is an important fact that this is implied by the Condition $[\delta_1, \delta_2]$. We formulate the contapositive of this implication in the following basic lemma:

Lemma 3.1

Suppose that the value $x_o \in S_\lambda \cap [\delta_1, \delta_2]$ is such that the operator $A(x_o)$ is not invertible on Z_n, the n-dimensional complex Euclidean space. Moreover, let a Z_n be a nonzero vector for which $A(x)a = 0$. Then the function

(3.8) $$g(x) = \frac{\lambda_1 k_1(x) a_1 + \ldots\ldots + \lambda_n k_n(x) a_n}{x - x_o}$$

is an eigenfunction of the operator M+K on $\mathcal{H} = (\alpha | [\delta_1, \delta_2], \alpha - 1, \lambda. \hat{\Omega})$ with eigenvalue x_o.

This lemma overlaps with a lemma of Friedrichs, [B1, a], concerning possibly non-selfadjoint operators K. It assumes more and it concludes more inasmuch as it assumes that K is selfadjoint and it concludes that the eigenfunction blows up near the exceptional value α_o, as a fractional power of the independent variable.

3.8

We start the proof of the lemma by recalling the selfadjoint version of the said lemma, which refers to the operator M+K on $\mathcal{B}(\alpha, \alpha-1, \lambda, \tilde{\Omega})$. More specifically on this space there are several operators which can be defined with the aid of the gentle kernel K(x,y). The operator entering the lemma is defined by

$$Kg(x) = \int K(x,y) \frac{\tilde{g}(y)}{x_o - y} d\lambda(y) + i\pi K(x,x_o)\tilde{g}(x_o) ,$$

for

$$g(y) = \frac{\tilde{g}(y)}{x_o - y} .$$

Since by definition $\mathcal{B}(\alpha, \alpha-1, \lambda, \tilde{\Omega})$ is the set of finite linear combinations of such functions, K is well defined on the entire space. Then using these notations we have the following which is essentially due to Friedrichs [Bla]:

<u>Lemma 3.2</u>

<u>Suppose that the value x_o is such that the operator $A(x_o)$ is not invertible on Z_n, moreover a is a nonzero vector for which $A(x_o)a = 0$. Then the function</u>

$$(3.8) \qquad g(x) = \frac{\lambda_1 k_1(x) a_1 + \cdots\cdots + \lambda_n k_n(x) a_n}{x - x_o}$$

<u>is an eigenfunction of M+K on $\mathcal{B}(\alpha, \alpha-1, \lambda, \tilde{\Omega})$ with eigenvalue x_o.</u>

Next we need an elementary fact on Hölder continuous functions.

3.9

According to Muskhelishvili, [El,b], if the function f is Hölder continuous with exponent α near x-x_o and $0 \leq \widetilde{\alpha} < \alpha$, then the function

$$g(x) = \frac{f(x) - f(x_o)}{(x-x_o)^{\widetilde{\alpha}}}$$

is Holder continuous near x_o with exponent $(\alpha - \widetilde{\alpha})$. From this fact, from the definition of the space $\mathcal{K}(\alpha \mid [\delta_1, \delta_2], \alpha-1, \widetilde{\delta})$, and from the previous lemma we conclude that in order to complete the proof of the basic lemma all that we have to establish is that the numerator of (3.8) vanishes at x_o. This is the statement of the next lemma:

<u>Lemma 3.3</u>

<u>Suppose that x_o is the exceptional value entering the two previous lemmas, and $A(x_o)\underline{a} = 0$. Then</u>

$$\lambda_1 k_1(x_o)a_1 + \ldots\ldots + \lambda_n k_n(x_o)a_n = 0.$$

Let $f(x)$ denote the expression on the left of this equation that is

$$(3.9) \qquad f(x) = \lambda_1 k_1(x)a_1 + \ldots\ldots + \lambda_n k_n(x)a_n \quad .$$

Then we maintain that this function is closely related to the imaginary part of the matrix $\Lambda A(x)$, where Λ is the matrix formed by the nonzero eigenvalues of K. More specifically we claim the following:

3.10

Proposition 3.1

Let the operator $C(x)$ be the imaginary part of Λ $A(x)$ and let the function $\mathfrak{f}(x)$ be defined for an arbitrary vector $\underline{a} \in Z_n$ by (3.9). Then the inner product $(\underline{a}, C(x)\underline{a})$, taken in Z_n, equals π-times the square of the norm of $\mathfrak{f}(x)$ taken in $\mathfrak{H}$, that is

$$(3.10) \qquad (\underline{a},C(x)\underline{a}) = \pi|\mathfrak{f}(x)|^2 \quad .$$

This statement is an immediate consequence of the defining equation of $A(x)$. For, we see from (3.7) that $C(x)$, the imaginary part of Λ $A(x)$, is given by the matrix

$$(3.11) \qquad C_{ij}(x) = \pi\lambda_i\lambda_j\ (k_i(x),\ k_j(x)) \quad .$$

Now clearly

$$\begin{aligned}(\underline{a},C(x)\underline{a} &= \pi \sum_i \bar{a}_i \sum_j \lambda_i\lambda_j\ (k_i(x),k_j(x))\ a_j \\ &= \pi \sum_{i,j} (\lambda_i k_i(x)a_i,\ \lambda_j k_j(x)a_j\ ) = \\ &= \pi|\lambda_1 k_1(x)a_1 + \dots + \lambda_n k_n(x)a_n|^2 = \pi|\mathfrak{f}(x)|^2\end{aligned}$$

and relation (3.10) is established.

Next we need a simple but important corollary of this proposition:

Proposition 3.2

The nullspace of the operator Λ $A(x)$ equals the intersection of the nullspaces of the real and imaginary parts of Λ $A(x)$.

3.11

Note that this statement is not true for arbitrary operators acting on Z_n, it depends on the specific form of $A(x)$, which in turn depends on the fact that the matrix $A(x)$ is associated with a self-adjoint perturbation K. On the other hand, once the statement is observed, its validity follows from the previous proposition and from the fact that the i^{th} component of the vector $C(x)\underline{a}$, $(C(x)\underline{a})i$, is given by the formula,

$$(C(x)\underline{a})_i = (\lambda_i k_i(x),\ f(x)\ )\quad .$$

Now from these propositions we can easily derive the statement of the lemma. For, by assumption $A(x_o)\underline{a} = 0$, hence $\bigwedge A(x_o)\underline{a} = 0$ and $(\underline{a},\ \bigwedge A(x_o)\underline{a}) = 0$. Now in view of the fact that the imaginary part of the leftside is $(\underline{a},\ C(x_o)\underline{a}\ )$ we conclude from Proposition 3.1 that $f(x_o) = 0$. This by definition means

$$\lambda_1 k_1(x_o)a_1 + \ldots\ldots + \lambda_n k_n(x_o)a_n = 0,$$

that is Lemma 3.3 is established, which in turn, establishes the basic Lemma 3.1 and Theorem 3.1.

4.1

Locally gentle perturbations of rank one.

Let K be a selfadjoint operator of rank one which is gentle over some interval $[\delta_1,\delta_2]$ which we assume to be bounded. In other words K is an operator of the form $K = \pm k \otimes k$, where the function k is gentle in $[\delta_1,\delta_2]$ in the sense of Section 2. For brevity we assume that the nonzero eigenvalue of K is one, that is $K = k \otimes k$ and $||k|| = 1$.

In this section we shall construct the spectral projectors of M+K over the interval $[\delta_1,\delta_2]$. First we shall show that the condition introduced in the previous section to ensure that the solution of the Friedrichs' equation is locally gentle also ensures the following: The disturbed resolvent can be continued weakly on a dense set onto the interval $[\delta_1,\delta_2]$, either from above or from below. Then we shall establish the important fact that the spectral projector of M+K over the interval $[\delta_1,\delta_2]$ can be expressed in terms of the solution of the Friedrichs' equation which is gentle over this interval.

We start this section with a simple propostion relating the disturbed and undisturbed resolvents.

Proposition.

Let R(z) denote the resolvent of $M+k \otimes k$ and let $R_o(z)$

denote the resolvent of M, that is $R(z) = (z-M-k \;\; x \;\; k)^{-1}$ and $R_o(z) = (z-M)^{-1}$. Then

$$(4.1) \quad R(z) = R_o(z) + \alpha^{-1}(z) R_o(z) k \otimes R_o^*(z) k \ ,$$

where

$$(4.2) \quad \alpha(z) = 1 - (k, R_o(z)k) = 1 - \int \frac{(k(y),k(y))}{z-y}\, d\lambda(y) \quad .$$

Next we turn to the proof of the earlier mentioned fact that the condition introduced in the previous section also ensures that R(z) can be continued weakly on a dense set onto $[\delta_1,\delta_2]$, either from above or from below. For this dense set we choose the set of those functions in $\mathcal{L}_2(\lambda, \mathfrak{H})$ which are gentle in $[\delta_1,\delta_2]$. Then we state the more specific lemma:

Lemma 4.1

Suppose that K is an operator of rank one which is gentle over $[\delta_1,\delta_2]$ moreover the operator M+K satisfies Condition $[\delta_1,\delta_2]$. Then for every pair of functions f,g, which are gentle in $[\delta_1,\delta_2]$, the family of complex valued functions,

$$(4.3) \qquad (g, R(x+i\varepsilon)f)$$

converges as $\varepsilon \to +0$. Moreover this convergence is uniform on the interval $[\delta_1,\delta_2]$.

Proof. According to the previous proposition

$$(4.4) \quad (g, R(x+i\varepsilon)f) = (g, R_o(x+i\varepsilon)f) + \\ + \alpha^{-1}(x+i\varepsilon)(g, R_o(x+i\varepsilon)k)(R_o^*(x+i\varepsilon)k,f) \quad .$$

4.3

Now for convenience of notation we assume that throughout this section x denotes a point of $[\delta_1,\delta_2]$. Then from the fact that the functions k,f, and g are gentle in the interval $[\delta_1,\delta_2]$ and from the Plemelj-Primalov theorem, [E1;a], we conclude that

$$(4.5)\quad \lim_{\varepsilon\to+0} (g, R_o(x+i\varepsilon)f) = \int \frac{(g(y),f(y))}{x-y} d\lambda(y)+i\pi(g(x),f(x)),$$

moreover that this limit is uniform in x. Next we maintain the existence of the uniform limit of the family of functions $\alpha^{-1}(x+i\varepsilon)$. For, from the definition of this function, that is fro (4.2) we conclude, as before, that

$$(4.6)\quad \lim_{\varepsilon\to o} \alpha(x+i\varepsilon) = 1 - \int \frac{(k(y),\ k(y))}{x-y} d\lambda(y)-i\pi(k(x),k(x)).$$

Now we observe that this function, which we shall denote by $\alpha(x+)$, can be identified with the coefficient function entering the equation (3.7) of the previous section. Hence according to Lemma 3.1 Condition $[\delta_1,\delta_2]$ ensures the non-vanishing of $\alpha(x+)$ in the interval $[\delta_1,\delta_2]$. This in turn, in view of the assumption that this is a bounded interval, ensures that this function remains bounded away from zero. Therefore the family of functions $\alpha^{-1}(x+i\varepsilon)$ tends uniformly to the function $\alpha^{-1}(x+)$ in the interval $[\delta_1,\delta_2]$, as ε tends to +0. This completes the proof of the lemma.

4.4

Finally we turn to the proof of the fact that the spectral projector of M+K over the interval $[\delta_1, \delta_2]$ can be expressed in terms of the solution of the Friedrichs' equation. More specifically, we claim the following:

Subtheorem.

Suppose that K is an operator of rank one which is gentle over $[\delta_1, \delta_2]$, moreover the operator M+K satisfies Condition $[\delta_1, \delta_2]$. Then E_δ the spectral projector of M+K over the interval $[\delta_1, \delta_2]$, can be written in the form

(4.7) $$E_\delta = (1 - \Gamma Q^*) C_\delta (1 + \Gamma Q).$$

Here the form Q satisfies the Friedrichs equation and is gentle over $[\delta_1, \delta_2]$. The operator C is defined by multiplication with the characteristic function of $[\delta_1, \delta_2]$.

The proof of the subtheorem is based on the "resolvent loop integral formula", which states the following, [E2,c]: for an arbitrary strictly selfadjoint operator with corresponding resolvent R(z) the spectral projector over the interval $[\delta_1, \delta_2]$ is given by

(4.8) $$E_\delta = \frac{1}{2\pi i} \oint R(z)dz ,$$

provided that the endpoints are not pointeigenvalues. Here the integral is to be taken as Cauchy principal value in the strong sense around a family of rectangles over the

interval $[\delta_1,\delta_2]$. Since the resolvent of a selfadjoint operator is analytic in the complex plane cut along the real axis, this family of contours can be deformed into the family of lines $[\delta_1+i\varepsilon,\delta_2+i\varepsilon]$, $[\delta_1-i\varepsilon,\delta_2-i\varepsilon]$, which in turn "can be carried on the interval" as follows:

$$(4.9)\qquad (f,(\frac{1}{2\pi i}\oint R(z)dz)g)$$
$$= \frac{1}{2\pi i}\lim_{\varepsilon\to +o}\int_{\delta_1}^{\delta_2}(f,(R(x+i\varepsilon)-R(x-i\varepsilon))g)dx\ .$$

The operator M is essentially selfadjoint and in view of the fact that K is selfadjoint and of rank one, M+K is also essentially selfadjoint. From this fact and from the fact according to the remark in Appendix III Condition $[\delta_1,\delta_2]$ implies that the endpoints are not pointeigenvalues of M+K, we conclude that formula (4.8) applies to this operator. Now choose the functions f and g to be gentle in $[\delta_1,\delta_2]$ and recall that according to Lemma 4.1 Condition $[\delta_1,\delta_2]$ ensures that the integrand in (4.9) admits a uniform limit. Insertion of this fact in (4.9) yields in view of (4.8) the relation

$$(4.10)\qquad (f,\ E_\delta\ g) = \frac{1}{2\pi i}\int_{\delta_1}^{\delta_2}[(f,R(x)g)]_-^+\ dx\ ,$$

where

$$[(f,R(x)g)]_-^+ = \lim_{\varepsilon\to +o}(f,R(x+i\varepsilon)g)-(f,R(x-i\varepsilon)g)\ .$$

Next recall that according to the proposition

$$(f,R(x+i\varepsilon)g) = (f,R_o(x+i\varepsilon)g)+\alpha^{-1}(x+i\varepsilon)\cdot$$
$$(4.11)\qquad \cdot(f,R_o(x+i\varepsilon)k)(R_o^*(x+i\varepsilon)k,g)\ .$$

4.6

Also recall the simple relation on the jump of an arbitrary triple product,

$$[\alpha\beta\gamma]_-^+ = [\alpha]_-^+\beta^+\gamma^+ + \alpha^-[\beta]_-^+\gamma^+ + \alpha^-\beta^-[\gamma]_-^+ \ .$$

On the otherhand using that the functions f and k are gentle in $[\delta_1,\delta_2]$ the Plemelj-Privalov theorem yields

$$(f,R_o(x \pm)\, k) = H_\pm \bar{f}k(x)$$

(4.12) and

$$[(f,R_o(x)k)]_-^+ = 2\pi i\ \bar{f}k(x) \ .$$

Therefore from (4.11) we obtain,

$$[(f,R(x)g)]_-^+ = 2\pi i\ \bar{f}g(x) + [\alpha^{-1}(x)]_-^+\ H_+\bar{f}k(x).\ \ H_+\bar{k}g(x)$$

(4.13)

$$+ \alpha^{-1}(x-)\cdot 2\pi i\ \bar{f}k(x).H_+g\bar{k}(x) + \alpha^{-1}(x-)H_-\bar{f}k(x).2\pi i\ \bar{k}g(x) \ .$$

Finally we complete the proof of the subtheorem by showing that this expression can be evaluated in terms of the solution of the Friedrichs' equation. This requires some algebraic manipulations. First we need the formula

$$[\alpha^{-1}(x)]_-^+ = 2\pi i\ \alpha^{-1}(x+)k(x)\ .\ \ \alpha^{-1}(x-)\bar{k}(x) \ ,$$

which is immediate from (4.2), the definition of $\alpha(z)$, and from (4.12). Next we need the fact noticed earlier that the function $\alpha(x+)$ can be identified with the coefficient function entering equation (3.7) . Therefore

(4.14) $\quad [\alpha^{-1}(x)]_-^+ = 2\pi i\ h\ \bar{h}(x)$,

4.7

where the function $h(x)$ is the solution of (3.7). Now for simplicity we assume that k is real; insertion of this fact and (4.14) in (4.13) yields

$$(4.15)\quad \frac{1}{2\pi i}\,[(f,R(x)g)]_-^+ = \bar{f}g(x) + hH_+\bar{k}\,g(x).\ \bar{h}H_+\bar{f}k(x) + + \bar{h}\,\bar{f}\,H_+\,\bar{k}g(x) + \bar{h}\,g\,H_-\,\bar{f}\,k(x)\quad .$$

Next consider the Γ transformation introduced in Section 2, which assigns a densely defined form to a form in $\mathfrak{G}_{aux}$. We maintain that for the particular form $h><k$ the form $C_\delta\Gamma(h><k)$ can be identified with an operator. For, according to (2.2)

$$<C_\delta f,\ \Gamma(h><k)g> = <f,\ C_\delta hH_+kg>\ ,$$

and since g is gentle over $[\delta_1,\delta_2]$, the function hH_+kg is in $\mathfrak{L}_2$, hence

$$\therefore\ <f,C_\delta hH_+kg> \;=\; (f,C_\delta hH_+kg).$$

That is the form $\Gamma(h><k)$ can be identified with the operator in the bracket, in short

$$(4.16)\qquad C_\delta\Gamma(h><k) = C_\delta hH_+k\ .$$

This statement is in accordance with Theorem 3.1, which states that under Condition $[\delta_1,\delta_2]$ the Friedrichs' equation admits a locally gentle solution, and with the fact that for a locally gentle form $\mathbb{Q}$ the form $C_\delta\Gamma G$ is bounded. Presently, however, we do not need the boundedness of $C_\delta\Gamma(h><k)$, all that we need

is that this operator is defined for locally gentle functions. Now we have seen that the operator hH_+k maps such functions into such functions and hence if x is a point in $[\delta_1,\delta_2]$, the function value $hH_+kg(x)$ is well defined. Combining this fact with relation (4.16) and remembering that according to our convention x always denotes a point in $[\delta_1,\delta_2]$, we have the following relations:

$$h\, H_+\, \bar{k}\, g(x) = \Gamma(h >\!< k)g(x)$$

(4.17)
$$\bar{h}\, H_+\bar{f}k(x) = \overline{h\, H_-\, \bar{k}\, f(x)} = \overline{\Gamma(h >\!< k)f + 2\pi i\, k\, \bar{h}\, \bar{f}(x)}$$

$$\bar{h}\, H_-\bar{f}k(x) = \overline{h\, H_+\, \bar{k}\, f(x)} = \overline{\Gamma(h >\!< k)f(x)}\ .$$

Hence insertion of (4.17) in (4.15) yields:

$$\frac{1}{2\pi i}\, [f,R(x)f)]^+_- = \bar{f}g(x) + \Gamma(h >\!< k)g(x)\cdot\overline{\Gamma(h >\!< k)f(x)}$$
$$+ 2\pi i\, k\, \bar{h}\, \bar{f}(x)\cdot\Gamma(h >\!< k)g(x)$$
$$+ \bar{h}\, \bar{f}(x)H_+\bar{k}g(x) + g(x)\cdot\overline{\Gamma(h >\!< k)f(x)}\ .$$

Now it is a matter of a simple algebra to derive from the definition of the function h the relation $(1+2\pi ikh)\bar{h} = h$, which holds in view of our simplifying assumption that k is real. From this we conclude

$$\frac{1}{2\pi i}\, [(f,\, R(x)g)]^+_- = \bar{f}\, g(x) + \overline{\Gamma(h >\!< k)f(x)}\cdot \Gamma(h >\!< k)g(x)$$

(4.18)
$$+ \bar{f}(x)\cdot\Gamma(h >\!< k)g(x) + \overline{\Gamma(h >\!< k)f(x)}\cdot g(x)\ .$$

Finally inserting this relation in (4.10), using that according

4.9

to the local gentleness propositions the operators $C_\delta\Gamma(h >< k)$ and $-\Gamma(k >< h)C_\delta$ are adjoints to each other we obtain

$$(f, E_\delta g) = (f, C_\delta) - (f, \Gamma(k >< h)C_\delta (h >< k)g)$$
$$+ (f, C_\delta \Gamma(h >< k)g) - (f, C_\delta\Gamma(k >< h)g) .$$

This in turn, in view of

$$(1-\Gamma Q^*)C_\delta(1+\Gamma Q) \doteq C_\delta - \Gamma Q^* C_\delta + C_\delta\Gamma Q - \Gamma Q^* C_\delta\Gamma Q$$

and $C_\delta\Gamma Q = \Gamma C_\delta Q = \Gamma(C_\delta h >< k)$, $\Gamma Q^* C_\delta = \Gamma Q^* C_\delta = -\Gamma(k >< C_\delta h)$

yields:

$$(f, E_\delta g) = (f, (1-\Gamma Q^*)C_\delta(1+\Gamma Q)g) .$$

Since the set of functions gentle in $[\delta_1, \delta_2]$ is dense in $\mathcal{L}_2(\lambda, \mathfrak{H})$ this relation establishes relation (4.7). In other words the subtheorem has been established.

5.1

Locally gentle perturbation of finite rank.

In this section we shall show that the previous statements on perturbations of rank one are typical. As a matter of fact, we shall show that if K is an operator of finite rank which is gentle over $[\delta_1 \delta_2]$ and the operator M + K satisfies Condition $[\delta_1 \delta_2]$, then the spectral transformation of the part of M + K over $[\delta_1 \delta_2]$ can be expressed in terms of the solution of the Friedrichs' equation.

We start with a lemma relating the perturbed and unperturbed resolvents. It is a corollary of a theorem of Fredholm, [E5], and was emphasized by Aronszajn and Weinstein.

Lemma 5.1

Let $K = \sum Kk_i \otimes k_i$, let z be a point in the resolvent set of M and set $(z - M)^{-1} = R_o(z)$. Then a necessary and sufficient condition for z to be in the resolvent set of M + K is that the matrix

$$(5.1) \quad A_{ij}(z) = \delta_{ij} - (Kk_i, R_o(z)k_j)$$

is invertible. Moreover in this case

$$(5.2) \quad R(z) = R_o(z) + \sum_{i,j} A^{-1}_{ij}(z) R_o(z)Kk_i \otimes R^*_o(z)k_j \, .$$

This lemma is an immediate consequence of the relation

(5.3) $R(z) = (z - M - K)^{-1} = R_o(z)\ (1 - K\ R_o(z))^{-1}$

which in turn is a consequence of the relation $(z - M - K) = (1 - K\ R_o(z))(z - M)$: For, according to the previously mentioned theorem of Fredholm [ ], the operator $(1 - K\ R_o(z))$ is invertible if an only if the matrix of the lemma is invertible and in this case $(1 - K\ R_o(z))^{-1} = 1 + \sum A_{ij}^{-1}(z)\ Kk_i \times R_o^*(z)k_j$. This in view of (5.3) establishes the lemma. Note that the matrix $A_{ij}(z)$ depends on the particular dyadic representation of K, but its determinant, the Fredholm determinant of $1 - K\ R_o(z)$, depends only on the operator, for it is the product of the eigenvalues of this operator.

Next we maintain that Lemma 4.1 of the previous section remains valid for operators of finite rank. More specifically we maintain the following:

Lemma 5.2

Suppose that K is an operator of finite rank which is gentle over $[\delta_1, \delta_2]$ furthermore the operator M + K satisfies Condition $[\delta_1 \delta_2]$. Then for every pair of functions f,g which are gentle in $[\delta_1 \delta_2]$, the family of complex valued functions,

5.3

$$(f, R(x + i\varepsilon)g)$$

<u>converges as $\varepsilon \to +0$. Moreover this convergence is uniform in x on the interval</u> $[\delta_1, \delta_2]$.

<u>Proof.</u> According to the previous lemma,

$$(f, R(x + i\varepsilon)g) = (f, R_o(x + i\varepsilon)g) + \sum_{i,j} A^{-1}_{ij}(x + i\varepsilon) \cdot$$
$$\cdot (f, R_o(x + i\varepsilon)Kk_i)(R^*_o(x + i\varepsilon)Kk_j, g) .$$

By assumption the operator K is gentle over $[\delta_1, \delta_2]$ hence the functions k_i are gentle in the interval $[\delta_1 \delta_2]$. From this fact and from the local gentleness of the functions f,g, we conclude, as in the previous section, that each of the functions $(f, R_o(x + i\varepsilon)Kk_j)$ admits a limit as $\varepsilon \to +0$, moreover that this limit is uniform x. (Recall that according to our convention x denotes a point of $[\delta_1 \delta_2]$). Next we maintain that this is also true about the family of functions $A^{-1}_{ij}(x + i\varepsilon)$. For, from the fact that the operator K is gentle over $[\delta_1 \delta_2]$ we conclude that

$$A(x + i\varepsilon) \to A(x+) \quad \text{as } \varepsilon \to +0 ,$$

moreover that this convergence is uniform in x. Now in analogy with the previous section we observe that the limit matrix A(x+) can be identified with the matrix (3.7) introduced earlier. Hence, according to Lemma 3.1

5.4

Condition $[\delta_1\delta_2]$ ensures that this matrix has an inverse and according to assumption we work with a bounded interval, hence the numerical valued function $|A(x+)|$ remains bounded away from zero. Therefore

$$A^{-1}(x+i\varepsilon) \to (A^{-1}(x+) \quad , \text{ as } \varepsilon \to +0$$

moreover this convergence is uniform in x. This establishes the lemma.

Next we shall show that the subtheorem of the previous section remains valid for operators of finite rank. First, however, we have to show that the "jump" of the family of functions $(f, R(x + i\varepsilon)g)$ can be evaluated in terms of the solution of the Friedrichs' equation, introduced in Section 3. More specifically, recalling the notation

$$[(f, R(x)g)]_-^+ = \lim_{\varepsilon \to +0} ((f, R(x + i\varepsilon)g) - (f, R(x - i\varepsilon)g)),$$

we maintain the following:

Lemma 5.3

Let Q be a locally gentle form which satisfies the Friedrichs' equation. Suppose that K is an operator of finite rank which is gentle over $[\delta_1\delta_2]$ and that M + K satisfies Condition $[\delta_1\delta_2]$. Then for any pair of functions f,g in $\mathfrak{L}_2(\lambda, \mathfrak{H})$ which in addition are gentle in $[\delta_1\delta_2]$,

5.6

On the otherhand, using the well known relation on the jump of a triple product, we obtain from (5.1),

$$\frac{1}{2\pi i}\,[(f,R(x)g)]_-^+ = \bar{f}g(x) + \sum_{i,j}[A_{ij}^{-1}(x)]_-^+ H_+\bar{f}\lambda_i k_i(x)\ .\ H_+\bar{k}_j g(x)\ .$$

For simplicity we assume that the functions $k_i(x)$ are real, then

$$\sum_i A_{ij}^{-1}(x-)\lambda_i k_i(x) = \lambda_i \overline{h_j(x)}\ .$$

Inserting this and (5.6) in (5.7) yields

$$\frac{1}{2\pi i}\,[(f,R(x)g)]_-^+ = \bar{f}\ g(x) + \sum_j \lambda_j h_j H_+\bar{k}_j g(x).\ \sum_i\ \overline{h_i}(x)H_+\lambda_i k_i\bar{f}(x)$$

(5.8)

$$+\sum_j \lambda_j\bar{h}_j\ \bar{f}\ H_+ g\ \bar{k}_j(x) + \sum_j \lambda_j\bar{h}_j g\ H_-\ \bar{f}\ \bar{k}_j(x)\ .$$

Now according to the definition of the operator Q and according to the definition of the Γ transformation we have

$$\sum_j h_j H_+\lambda_j\bar{k}_j\ g(x) = \Gamma Qg(x)$$

$$\sum_j \lambda_j\bar{h}_j H_+ k_j\bar{f}(x) = \overline{\sum_j \lambda_j h_j H_- k_j\bar{f}(x)} = \overline{\Gamma Qf(x)} + 2\pi i\sum \lambda_j k_j\bar{h}_j\bar{f}(x)$$

$$\sum_j \lambda_j\bar{h}_j H_-\bar{f}\ k_j(x) = \overline{\sum_j \lambda_j h_j\ H_+\ f\ \bar{k}_j(x)} = \overline{\Gamma Qf(x)}$$

Insertion of these relations in (5.8) and using that according

5.5

$$\frac{1}{2\pi i}[(f, R(x)g)]_-^+ = \overline{f}\, g(x) + \overline{\Gamma\, Qf(x).}\quad \Gamma\, Qg(x) +$$

$$(5.4)\qquad + \overline{f}\, \Gamma Qg(x) + \overline{\Gamma\, Qf(x).}\; g(x) .$$

This relation is the generalization of the previous relation (4.17). Its proof is somewhat tedious but it was the purpose of the present lemma to isolate the tedious part of the proof of a subtheorem, which we shall state later.

We start the proof of the lemma by recalling some facts and definitions. First we recall from Section 4 that

$$(f, R_o(x)k)^{\pm} = H_{\pm}\, \overline{f}\, k(x),$$

hence

$$[(f, R_o(x)k)]_-^+ = 2\pi i\; f\, \overline{k}(x) .$$

Then we recall from Section 3 that by definition

(3.7) $\quad h(x) = A^{-1}(x+\ \ k(x)$,

where we made use of the fact that the coefficient matrix of Section 3 can be identified with our present matrix $A(x+\ )$. From the last two relations in turn, using the definition of $A(x+\ )$ we obtain,

(5.5) $\quad [A(x)]_-^+ = -2\pi i\; k(x) \otimes \wedge\; k(x)$

and

(5.6) $\quad [A^{-1}(x)_-^+ = 2\pi i\; h(x) \otimes \wedge\; h(x)$.

to our simplifying assumption the functions k_i are real, yields:

$$\frac{1}{2\pi i}[(f, R(x)g)]_-^+ = \bar{f}\, g(x) + \Gamma Qg(x) \cdot \overline{\Gamma Qf(x)} + 2\pi i\, \Gamma Qg(x) \cdot$$

$$\cdot \sum_j \lambda_j k_j \bar{h}_j \bar{f}(x) + \sum \lambda_i \bar{h}_i \bar{f} H_+ \bar{k}_i g(x) + g(x) \cdot \overline{\Gamma Qf(x)} \ .$$

Next we maintain that if the functions $k_i(x)$ are real, then

$$(5.10) \qquad [1 + 2\pi i\, h \otimes \wedge\, k]\, \bar{h} = h \ .$$

For from relation (5.5) and from the definition of the vector h, that is from (3.7) we conclude,

$$A(x) - A(x+) = [A(x+)\, h(x) \otimes \wedge\, k(x)] \ .$$

According to Lemma 3.1 Condition $[\delta_1, \delta_2]$, ensures that for x in $[\delta_1, \delta_2]$ the matrix $A(x+)$ admits an inverse. Hence

$$A^{-1}(x+) A(x-) \bar{h}(x) = [1 + 2\pi i\, h(x) \otimes \wedge\, k(x)] \bar{h}(x) \ .$$

On the other hand, since k is real,

$$A^{-1}(x+) A(x-) \bar{h}(x) = A^{-1}(x+) k(x) = h(x) \ ,$$

which establishes (5.10). Now, in view of this relation we have

$$2\pi i\, \Gamma Qg(x) \sum_j \lambda_j k_j \bar{h}_j f(x) + \sum_i \lambda_i \bar{h}_i \bar{f}\, H_+ k_i g(x)$$

$$= f(x) \cdot \sum_i (h_i \sum_j \lambda_j k_j \bar{h}_j + \bar{h}_i) \lambda_i H_+ k_i g(x)$$

$$= f(x) \sum \lambda_i H_+ k_i g(x) = f(x) \cdot \Gamma Qg(x) \ .$$

Insertion of this in (5.9) yields

5.8

$$\frac{1}{2\pi i}[(f,\ R(x)g)]_{-}^{+} = \bar{f}\ g(x) + \Gamma Qg(x) \cdot \Gamma Qf(x) + f(x)\Gamma Qg(x) + g(x) \cdot \Gamma Qf(x)\ ,$$

which establishes the lemma.

Now let us recall the resolvent loop integral formula which states the following: The spectral projector of a strictly selfadjoint operator with resolvent $R(z)$ over the interval $[\delta_1, \delta_2]$ whose endpoints aren't pointeigenvalues, is given by:

$$(f,\ Eg) = \frac{1}{2\pi i} \lim_{\varepsilon \to +0} \int_{\delta_1}^{\delta_2} ((f,\ R(x+i\varepsilon)g) - (f, R(x-i\varepsilon)g))dx\ .$$

The operator M is essentially selfadjoint and in view of the fact that K is of finite rank so is M+K. Hence this formula applies to the operator. On the other hand, if M+K satisfies the conditions of Lemma 5.1, then the above family of functions admits a uniform limit. Therefore,

$$(f,\ Eg) = \frac{1}{2\pi i} \int_{\delta_1}^{\delta_2} [(f,\ R(x)g)]_{-}^{+} dx\ .$$

Now in Lemma 5.3 we evaluated the integrand in terms of the solution of the Friedrichs' equation. Inserting this expression in the integral, that is inserting relation (5.4), yields the following:

5.9

Subtheorem 5.1

Suppose that K is an operator of finite rank which is gentle over $[\delta_1,\delta_2]$ furthermore the operator M+K satisfies Condition $[\delta_1,\delta_2]$. Then E_δ,the spectral projector of M+K over the interval $[\delta_1,\delta_2]$ can be written in the form

(5.11) $$E_\delta = (1-\Gamma Q^*)C_\delta(1+\Gamma Q) ,$$

where Q is the solution of the Friedrichs equation.

Next we introduce the transformation

(5.12) $$U_\delta = C_\delta(1+\Gamma Q)$$

entering the subtheorem. We recall that $(M+K)_\delta$, denotes the part of M+K over the interval $[\delta_1,\delta_2]$, and the operator M_δ is defined similarly. Note that M_δ is the multiplication operator on $\mathcal{L}_2(\lambda/[\delta_1,\delta_2], \mathfrak{H})$, where $\lambda/[\delta_1,\delta_2]$ denotes the restriction of the measure λ to $[\delta_1,\delta_2]$.

Now from the previous subtheorem and from the product proposition of Section 2 we shall derive that the above U_δ is a spectraltransformation of $(M+K)_\delta$. More specifically we shall derive the following:

Theorem 5.1

Suppose that K is an operator of finite rank which is gentle over $[\delta_1,\delta_2]$, furthermore the operator M+K satisfies Condition $[\delta_1,\delta_2]$. Then the operator $(M+K)_\delta$ is unitarily

equivalent to M_δ via a transformation of the form $U_\delta = C_\delta(1+\Gamma Q)$, in particular, it is absolutely continuous. In other words U_δ is a partial isometry whose initial set is the eigenspace of M+K over $[\delta_1,\delta_2]$, and whose final set is $\mathcal{L}_2(\lambda|[\delta_1,\delta_2],\tilde{\eta})$, furthermore

(5.13) $\qquad (M+K)_\delta = U_\delta^* M_\delta U_\delta \quad .$

Proof. Let $[\lambda_i,\lambda_{i+1}]$ be an arbitrary finite family of subintervals of $[\delta_1,\delta_2]$. Then from the previous subtheorem we conclude the relation;

$$\sum \lambda_i E_i = U_\delta^* \left(\sum \lambda_i C_i \right) U_\delta \quad ,$$

where E_i is the spectral projector of M+K over $[\lambda_i,\lambda_{i+1}]$, and C_i is multiplication by this characteristic function. Now let us choose these subintervals in such a way that their union is $[\delta_1,\delta_2]$, and $\max(\lambda_{i+1} - \lambda_i) \to 0$. Then according to the spectral theorem for selfadjoint operators, [E5]:

$$\sum \lambda_i E_i \to E_\delta (M+K) E_\delta = (M+K)_\delta \quad ,$$

where E_δ is the spectral projector of M+K over $[\delta_1,\delta_2]$. Therefore

$$U_\delta^* \left(\sum \lambda_i C_i \right) U_\delta \to (M+K)_\delta \quad .$$

On the other hand, evidently

$$U_\delta^* \left(\sum \lambda_i C_i \right) U_\delta \to U_\delta^* M_\delta U_\delta \quad .$$

From these two relations we conclude the validity of relation (5.13) of the theorem.

Next we turn to the proof of the partial isometric character of U_δ. More specifically we shall establish the following:

<u>Lemma 5.4</u>

<u>The transformation $U_\delta = C_\delta (1+\Gamma Q)$ is a partial isometry whose initial set is $\mathcal{E}_\delta$, the eigenspace of M+K over $[\delta_1, \delta_2]$ and whose final set is $\mathcal{L}_2(\lambda/[\delta_1,\delta_2], \mathfrak{H})$.</u>

First we maintain that the adjoint of the transformation U_δ, that is the transformation U_δ^*, is a partial isomety with initial set $\mathcal{L}_2(\lambda|[\delta_1,\delta_2], \mathfrak{H})$. In other words we maintain the validity of the relation

(5.14) $\quad U_\delta U_\delta^* = C_\delta$.

For the case of gentle operators, that is for the case of an interval $[\delta_1, \delta_2]$ which contains S_λ, the support of the measure λ, this was established by Friedrichs. The present statement can be established similarly by replacing the gentleness axioms by the corresponding local version. Nevertheless, for the sake of completeness let us carry out the details:

5.12

According to (5.12) the definition of U_δ,

$$U_\delta U_\delta^* = C(1+\Gamma Q)(1-\Gamma Q^*)C = C + C\Gamma Q\Gamma Q^*C + C\Gamma QC - C\Gamma Q^*C$$

Now the product proposition of Section 2 yields

$$C\Gamma Q\Gamma Q^*C = \Gamma(CQ\Gamma Q^*C + C\Gamma(Q)Q^*C) \, ,$$

which in turn inserted in the previous relation yields

$$(5.15) \quad U_\delta U_\delta^* = C + \Gamma(CQ(1 - \Gamma Q^*)C - C(1+ \Gamma Q)Q^*C) \, .$$

From the fact that Q is a solution of the Friedrichs equation in the interval $[\delta_1,\delta_2]$ and from the selfadjoint character of K we infer that the operator $CQ(1 - \Gamma Q^*)C$ is selfadjoint. Hence the operator in the brackets of (5.15) is zero and therefore

$$U_\delta U_\delta^* = C \, ,$$

that is the relation (5.14) has been established. In other words our statement that U_δ^* is a partial isometry with initial set $\mathcal{L}_2(\lambda|[\delta_1,\delta_2], \mathfrak{H})$ has been established.

Now it is a general operator theoretic fact that the adjoint of a partial isometry is a partial isometry whose final set is the initial set of the original isometry. Applying this to the partial isometry U_δ^* we obtain that U_δ is a partial isometry whose final set is $\mathcal{L}_2(\lambda[\delta_1,\delta_2], \mathfrak{H})$.

5.13

Since the operator $U_\delta^* U_\delta$ is the projector on the initial set of U_δ and according to the subtheorem

$$U_\delta^* \, U_\delta = E_\delta \ ,$$

we see that the initial set of U_δ is $\mathcal{E}_\delta$ the eigenspace of M+K over the interval $[\delta_1, \delta_2]$. This completes the proof of the lemma, which in turn completes the proof of the basic Theorem 5.1.

Next we formulate a sharper version of the basic theorem. In doing this we shall assume that the pointeigenvalue of the operator M+K on $\mathcal{B}(\alpha/[\delta_1,\delta_2],\alpha-1,\lambda,\Gamma)$ are disjoint only from the open interval (δ_1,δ_2), to which we shall refer shortly as Condition (δ_1,δ_2). We shall show that in this case the part of M+K over the open interval (δ_1,δ_2), which we denote again by $(M+K)_\delta$, is unitarily equivalent to M_δ. More specifically we shall establish the following:

Theorem 5.2

Suppose that K is a selfadjoint operator of finite rank which is gentle over every closed subinterval of (δ_1,δ_2), furthermore the operator M+K satisfies Condition (δ_1,δ_2). Then the part of M+K over the interval (δ_1,δ_2), $(M+K)_\delta$, is unitarily equivalent to M_δ via a transformation of the form $C_\delta(1+\Gamma Q$, in particular it is absolutely continuous. Here Q is a

5.14

<u>locally gentle solution of the Friedrichs' equation.</u>

Before establishing this theorem, let us note that under Condition $[\delta_1,\delta_2]$ its conclusion implies the conclusion of the basic theorem. For, from Condition $[\delta_1,\delta_2]$ we conclude that the set of pointeigenvalues of M+K on $\mathcal{L}_2(\lambda, \widetilde{\mathcal{H}})$ is disjoint from the closed interval $[\delta_1,\delta_2]$. Hence the endpoints, in particular, are not pointeigenvalues and therefore the spectral projector over the open interval (δ_1,δ_2) equals the spectral projector over the closed interval $[\delta_1,\delta_2]$.

Next we maintain that this theorem can be easily derived from the basic theorem. For, let $\delta_i = [\lambda_i, \lambda_{i+i}]$ be a family of increasing subintervals of $\delta = (\delta_1,\delta_2)$ whose union is (δ_1,δ_2). Then, evidently the basic theorem applies to the part of M+K over $\delta_i = [\lambda_i, \lambda_{i+i}]$. Hence if Q is the solution of the Friedrichs equation, and C_i is multiplication by the characteristic function of δ_i, then the part of M+K over δ_i is unitarily equivalent to the corresponding part of M via the transformation $C_i + C_i \Gamma Q$.
In other words,

5.15

$M+K = U_i^* MU_i$ holds on $E[\delta_i] \quad = \quad_i$,
with $U_i = C_i + C_i | Q$,

for every interval δ_i. Hence

$$M+K = U^* MU \text{ on } \mathcal{E}_i ,$$

(5.16)

$$\text{with } U = C + C\Gamma Q .$$

Now according to the general theory of linear operators, [E4], the spectral projectors of a strictly selfadjoint operator define a projector valued measure which is countably additive in the strong topology. From this fact and from the fact that δ_i is a monotone sequence whose union is δ we conclude that

$$E[\delta_i] \to E(\delta) .$$

From this in turn we conclude that

$$\overline{\bigcup \mathcal{E}_i} = \overline{\bigcup E[\delta\ ]} \tilde{\mathfrak{H}} = E(\delta) \tilde{\mathfrak{H}} = \mathcal{E}_\delta$$

where bar denotes closure. Therefore, in view of (5.16)

$$M+K = U^* MU \text{ on } _\delta$$

that is the part of M+K over δ is unitarily equivalent to the corresponding part of M, via the transformation $U = C + \Gamma CQ$. This establishes theorem 5.2.

Next we maintain that in this theorem Condition $[\delta_1, \delta_2]$ can be replaced by a condition on the Hölder

5.16

exponent entering the definition of our space of locally gentle forms. More specifically suppose that $\alpha > \frac{1}{2}$ in $\mathcal{G}(\alpha, \lambda | [\delta_1, \delta_2], \tilde{\mathfrak{H}})$. Then in view of this choice of α we have

$$\mathcal{B}(\alpha/[\delta_1, \delta_2], \alpha - 1, \lambda, \mathfrak{H}) \quad \mathcal{L}_2(\lambda, \mathfrak{H}) ,$$

and thus the pointeigenvalues of M+K on $\mathcal{B}$ are also pointeigenvalues of M+K on $\mathcal{L}_2$. From this fact, from the selfadjoint character of K and from the separability of $\mathcal{L}_2$ we conclude that the set of pointeigenvalues of M+K on $\mathcal{B}$ is countable. Note that since $\mathcal{B}$ is not separable, this statement is not obvious. On the other hand, we claim that the intersection of the set of pointeigenvalues of M+K on $\mathcal{B}$ with the interval $[\delta_1, \delta_2]$ is a closed set. For, according to an extended version of the basic Lemma 3.1, given in Appendix III, this set is the set of zeros of det $A(x)$ in $[\delta_1, \delta_2]$. Since the matrix elements of $A(x)$ are continuous in $[\delta_1, \delta_2]$ the intersection of the set of zeros of det $A(x)$ with the interval $[\delta_1, \delta_2]$ is a closed set. This establishes the claim and the statement that the intersection of the set of pointeigenvalues of M+K on $\mathcal{B}$ with the interval $[\delta_1, \delta_2]$ is a <u>countable closed</u> set. Therefore, the complement of this

set with reference to the interval $[\delta_1,\delta_2]$ is an open set, moreover, it can be written as the union of disjoint open intervals in such a way that the endpoints of these intervals are pointeigenvalues of M+K. Let the family of these intervals be denoted by $\{(\lambda_i, \lambda_{i+i})\}$. Then evidently on each of these open intervals Condition (λ_i,λ_{i+}) is satisfied and hence from Theorem 5.2 we can conclude that <u>the part of M+K over the open interval</u> $(\lambda_i,\lambda_{i+i})$ <u>is unitarily equivalent to the corresponding part of M</u> via a transformation of the form $C_i + C_i \Gamma Q$, where Q is the solution of the Friedrichs equation. Note that Q and ΓQ are forms, nevertheless the form $C_i \cdot (1+\Gamma Q)$, can be identified with a bounded operator. Finally, combining this statement with the countable additivity of the spectral projectors a limiting argument, for example the one entering the proof of Theorem 5.2, yields the following:

<u>Theorem 5.3</u>

<u>Suppose that K is a selfadjoint operator of finite rank in</u> $\mathfrak{G}(\alpha/[\delta_1\delta_2], \lambda, \mathfrak{H})$ <u>with</u> $\alpha > \frac{1}{2}$. <u>Then the continuous part of M+K over</u> $[\delta_1,\delta_2]$ <u>is unitarily equivalent to the part of</u> M <u>over</u> $[\delta_1,\delta_2]$ <u>via a transformation of the form</u> $C_\delta(1+\Gamma Q)$.

5.18

<u>Here the form Q is the solution of the Friedrichs' equation and</u> C <u>is the operator of multiplication by the characteristic function of</u> $[\delta_1, \delta_2]$. <u>In particular, the continuous part of</u> M+K <u>over</u> $[\delta_1, \delta_2]$ <u>is absolutely continuous.</u>

Note that this theorem gives no information on the discrete part of M+K over $[\delta_1, \delta_2]$, that is it may be of finite rank or it may be of infinite rank. According to a verbal communication of C.C. Conley it is possible that a locally gentle perturbation of finite rank produces infinitely many pointeigenvalues.

<u>Remark.</u>

Onc can slightly relax the assumptions of this theorem inasmuch as it suffices to assume that K is a selfadjoint operator of finite rank which is in $\mathfrak{G}(\alpha[\lambda_i, \lambda_{i+1}], \lambda, \tilde{\mathfrak{I}})$ for every closed proper subinterval $[\lambda_i, \lambda_{i+1}]$ of $[\delta_1, \delta_2]$ with $\alpha > 1/2$. Then using the same limiting procedure as before, we see that the continuous part of M+K over the interval (δ_1, δ_2) is unitarily equivalent to the corresponding part of M. An interesting special case arises if (δ_1, δ_2) is the interior of S_λ, the support of the undisturbed spectral measure. For, in this case, from Weyl's theorem on compact perturbations, we conclude that the continuous part of M+K over (δ_1, δ_2) equals the entire continuous part of M+K.

§ 6

Other perturbations.

A theorem of Friedrichs, [B2], treats perturbations which are small in some gentleness norm and the theorems of the previous section treat locally gentle perturbations of finite rank. In this section we shall show that these theorems can be combined to treat perturbations which can be written as the sum of two such perturbations. First, however, we make a digression and describe the notion of a " completely gentle operator" .

If K is a gentle integral operator whose kernel K(x,y) is a compact operator on the accessory space $\mathfrak{H}$, for every fixed value (x,y), then K is called completely gentle, [B7]. This is the condition introduced by L.D. Faddeev and O.A. Ladyssenskaia. Next let $\mathfrak{G}(\alpha,\beta,\lambda, \mathfrak{H})$ denote the gentle space of operators introduced by J. Schwartz in connection with " some non-selfadjoint operators", [B5]. We observed in the previous paper that a completely gentle operator in this space is of almost finite rank in any intermediate space $\mathfrak{G}(\tilde{\alpha},\tilde{\beta},\lambda, \mathfrak{H})$, with $0 \leq \tilde{\alpha} < \alpha$, $0 \leq \tilde{\beta} < \beta$. That is a completely gentle operator in $\mathfrak{G}(\alpha,\beta,\lambda, \mathfrak{H})$ can be approximated by operators of finite rank in the norm of the intermediate space. One is tempted to conclude from this fact that a gentle operator is compact. This is not justified, however, since compact operators are of almost finite rank with reference to the operator norm and completely gentle operators are of almost finite rank with reference to a gentleness norm. As a matter of fact the examples of the next section show that an operator with continuous spectrum may be gentle with respect to the operator $-D^2$.

After this digression we return to those operators which can be written as the sum of an operator which is small in some gentleness norm and a locally gentle operator of finite rank. To be sure the class of such operators is rather restrictive, for all that we know is that the sum of a completely gentle operator and a locally gentle operator of finite rank is in this class. Nevertheless in the next section we shall show that this class contains some operators of Mathematical Physics.

Next we combine the Friedrichs theorem on " small perturbations" with the theorems of the previous section. We start with Theorem 5.3 which treats locally gentle perturbations of finite rank with Hölder exponent $\alpha > \frac{1}{2}$. We maintain that this combination leads to the following:

Theorem 6.1

Suppose that the operator K is of the form $K = G+F$, where G is a completely gentle operator in $\mathfrak{G}(\alpha,\beta,\lambda,\Gamma)$ and F is a locally gentle operator of finite rank in $\mathfrak{G}(\alpha|[\delta_1\delta_2],\lambda,\Gamma)$ with $\alpha > \frac{1}{2}$. Then the operator M+K is essentially selfadjoint. The continuous part of M+K over $[\delta_1,\delta_2]$, $(M+K)_{c,\delta}$ is unitarily equivalent to M_δ, in particular it is absolutely continuous. Furthermore, $(M+K)_{c,\delta}$ admits a spectral transformation of the form U_1U_2, where $U_1 = 1 + \Gamma Q_1$ where Q_1 belongs to some intermediate gentle space and $U_2 = C_\delta + C_\delta\Gamma Q_2$ where Q_2 is a locally gentle form.

Proof. It was noted previously that a completely gentle operator

in $\mathfrak{S}(\alpha,\beta,\lambda,\Gamma)$ is of almost finite rank in any intermediate space $\mathfrak{S}(\tilde{\alpha},\tilde{\beta},\lambda,\Gamma)$. Hence G can be written as $G_s + G_f$ where the norm of G_s in the intermediate space is less than one and G_f is of finite rank. Hence from the theorem on small perturbations, we can conclude that $M+G_s$ is unitarily equivalent to M via a transformation of the form $1 + \Gamma Q_1$. More specifically

$$M + G_s = U_1^* MU_1 \quad ,$$

(6.1) $$U_1 = 1 + \Gamma Q_1 \; , \quad Q_1 \in \mathfrak{S}(\tilde{\alpha},\tilde{\beta},\lambda,\Gamma) \; .$$

From this in turn we conclude

(6.2) $$(M+K) = U_1^* \, [M + U_1(G_f + F)U_1^*]U_1 \quad .$$

Now the operator $U_1(G_f+F)U_1^*$ is of finite rank hence the operator in the brackets is essentially selfadjoint and since this notion is unitarily invariant so is the operator on the left. This establishes the first statement of the theorem and thus the spectral projectors of M+K are well defined.

Next consider, $(M+K)_{c,\delta}$, the continuous part of M+K over the interval $[\delta_1,\delta_2]$. This operator, according to (6.2) is unitarily equivalent to the corresponding part of the operator in brackets, that is

(6.3) $$(M+K)_{c,\delta} = U_1^*[M+U_1(G_f + F)U_1^*]_{c,\delta} \, U_1 \quad .$$

We maintain that Theorem 5.3 applies to the operator in the brackets. In other words we maintain that the operator $U_1(G_f+ F)U_1^*$ is of finite rank and it is gentle over the interval $[\delta_1,\delta_2]$ with exponent greater than 1/2. For, if $\{f_i\}$ are the eigenfunctions

of $G_f + F$ then clearly $\{U_1 f_i\}$ are the eigenfunctions of this operator. Then we need the following elementary but important fact on gentle operators: <u>If Q is in $\mathfrak{G}(\tilde{\alpha},\tilde{\beta},\lambda,\tilde{\Pi})$ and f is in $\mathfrak{G}(\alpha/[\delta_1,\delta_2],\lambda,\Pi)$ and $\tilde{\alpha} < \alpha$ then ΓQf is in $\mathfrak{G}(\tilde{\alpha}/[\delta_1,\delta_2],\lambda,\Pi)$.</u>

Now so far $\tilde{\alpha} < \alpha$ was arbitrary and by assumption $\alpha > \frac{1}{2}$, hence we may choose $\tilde{\alpha} > 1/2$. These facts combined with (6.1) according to which $U_1 = 1 + \Gamma Q_1$ establish the local gentleness of $U_1(G_f + F)U_1^*$ with exponent $\tilde{\alpha} > \frac{1}{2}$. Therefore Theorem 5.3 applies to the operator in the brackets of (6.3) and yields

$$[M + U_1(G_f + F)U_1^*]_{c,\delta} = U_2^* M U_2$$

(6.4)
$$U_2 = C_\delta + C_\delta \Gamma Q_2 ,$$

where Q_2 is locally gentle. Inserting this relation in (6.3) we obtain that $(M+K)_{c,\delta}$ is unitarily equivalent to the corresponding part of M via the transformation U_2U_1, which establishes the theorem.

<u>Remark</u>. One can give a slightly sharper formulation of this theorem with the aid of a local version of the space $\mathfrak{G}(\alpha,\beta,\lambda,\Pi)$ defined as follows: Let $[\delta_1,\delta_2]$ be a bounded interval and set

$$\Delta_{h_1}K(x,y) = (K(x+h_1,y) - K(x-h_1,y))(2h_1)^{-\alpha}$$

$$\Delta_{h_2}K(x,y) = (K(x,y+h_2) - K(x,y-h_2))(2h_2)^{-\alpha}$$

for $x,y \in [\delta_1,\delta_2]$.

Next define

$$|K|_\delta = \sup_{x,y\in[\delta_1,\delta_2]} |K(x,y)|$$

and

$$||K||_{\alpha,\delta,\kappa} = \max\left\{\sup_{0\le h_1\le\kappa} |\Delta_{h_1}K|_\delta,\ \sup_{0\le h_2\le\kappa}|\Delta_{h_2}K|_\delta,\ \sup_{\substack{0\le h_1\le\kappa\\0\le h_2\le\kappa}}|\Delta_{h_1}\Delta_{h_2}K|_\delta\right\}.$$

Finally define $\mathfrak{S}(\alpha,\lambda|[\delta_1,\delta_2],\ \mathfrak{H})$, the space of operators gentle over the interval $[\delta_1,\delta_2]$ with exponent α, to be the set of those operators K to which there is a number κ and an interval $[\tilde\delta_1,\tilde\delta_2] \supset [\delta_1,\delta_2]$, such that

(6.5) $$||K||_{\alpha,\tilde\delta,\kappa} < \infty \quad .$$

Note that in case K is of finite rank this definition coincides with the previous one.

Next we maintain that the conclusion of Theorem 6.1 remains valid if G is a completely gentle operator in $\mathfrak{S}(\alpha,\beta,\lambda,\ \mathfrak{H})$ with arbitrary positive exponent α and G and F are gentle over every closed subinterval of $[\delta_1,\delta_2]$ with exponent greater than 1/2. For, from the fact that the Friedrichs theorem on small perturbations holds for arbitrary positive exponents, we conclude, as before, the validity of relation (6.3). Then from the assumption that G and K are gentle over every subinterval with exponent greater than 1/2, we conclude that so is $U_1(G_f+F)U_1^*$. From, this in turn using the remark after Theorem 5.3, we conclude the validity of relation (6.4) for the present operator, and the validity of the remark.

Finally we combine Theorem 5.2 with the theorem on " small perturbations" , and maintain that it yields the following:

Theorem 6.2

Let K be an operator of the form $K = G+F$ where G is a completely gentle operator in $\mathfrak{G}(\alpha,\beta,\lambda,\mathfrak{H})$ and F is an operator of finite rank in $\mathfrak{G}(\alpha/[\delta_1,\delta_2],\lambda,\mathfrak{H})$. Suppose that the operator M+K satisfies Condition $[\delta_1,\delta_2]$. Then M+K is essentially selfadjoint. The part of M+K over $[\delta_1\delta_2]$, $(M+K)_\delta$ is unitarily equivalent to M_δ, in particular it is absolutely continuous. Furthermore $(M+K)_\delta$ admits a spectral transformation of the form U_1U_2, where $U_1 = 1+\Gamma Q_1$ with $Q_1 \in \mathfrak{G}(\tilde{\alpha},\tilde{\beta},\lambda,\mathfrak{H})$ and $U_2 = C_\delta + C_\delta \Gamma Q_2$ where Q_2 is locally gentle.

The proof of this theorem is similar to the proof of the previous theorem. For, brevity, let us not carry out the details, just mention the following relevant fact: the transformation $U_1 = 1 + \Gamma Q_1$ maps $\mathfrak{P}(\alpha,/[\delta_1,\delta_2], 1-\alpha, \lambda, \mathfrak{H})$ into $\mathfrak{P}(\alpha/[\delta_1,\delta_2],1-\alpha, \mathfrak{H})$, which is a consequence of Lemma 3 of Appendix IV.

7.1

APPLICATIONS

First we take as undisturbed operator, $-\Delta$, the negative Laplacian in free space. More specifically let R_n denote the n-dimensional Euclidean space, where $n \geq 1$, and let $\mathcal{D}$ denote the set of infinitely differentiable functions with compact support. Then we shall denote by $-\Delta$, the closure of the Laplacian on $\mathcal{D}$ in $\mathcal{L}_2(R_n)$. As disturbance we take a selfadjoint multiplication type operator, or potential operator that is an operator of the form

$$M_q f(x) = q(x) f(x) \quad ,$$

where $q(x)$ is a real function. Then the operator $-\Delta + M_q$ is the Schrodinger operator associated with a systems of particles in a potential field and it has been investigated by several authors from several point of views. [D,F]

For the case of dimension one, conditions on the function q were given by Titchmarsh, [D7], which ensured that the continuous part of $-\Delta + M_q$ was unitarily equivalent to $-\Delta$. For the case of dimension 3, conditions on the function q were given by Povzner, [D2], which ensured the above conclusion. Moreover, he described the spectral transformation of this operator. Later Povzner results were generalized by T. Ikebe, who also indicated how to carry them over to dimension $n \geq 2$.

7.2

Thus the spectral theory of the operator $-\Delta + M_q$ is well known, for a large class of ''potentials'' q.

Next we shall illustrate how our abstract Theorem 6.1 can be applied to the operator $-\Delta + M_q$. The results which we shall obtain will be weaker than the corresponding ones of Titchmarsh and Povzner-Ikebe, inasmuch as Condition 7.1, to be introduced, is much more restrictive than the condition of these authors. On the other hand, the approach via gentleness has the interesting feature, that there is no need to distinguish between cases of even and odd number of space dimensions, although the one dimensional case has to be considered separately. For, we shall see that under Condition 7.1, to be stated later, the operator M_q is <u>locally gentle</u> with respect to $-\Delta$ in case of dimension $n = 1$ and it is <u>gentle</u> in case of dimension $n \geq 2$.

<u>A. Schrodinger operator in free space of dimension one.</u>

It is well known that the spectral measure of $-\Delta$ is equivalent to λ, the restriction of the Lebesque measure to $(0,\infty)$, and that the spectral multiplicity of $-\Delta$ is two. Moreover, [D7], a spectral transformation of $-\Delta$, corresponding to the measure λ, is given by an integral transformation U with the vector valued kernel,

7.3

$$U(x,y) = \frac{1}{\sqrt{2\pi}} x^{-1/4} \begin{pmatrix} \cos \sqrt{x}\, y \\ \sin \sqrt{x}\, y \end{pmatrix} .$$

Note that U maps $\mathcal{L}_2(-\infty,+\infty)$ onto $\mathcal{L}_2(\lambda,Z_2)$ and the inverse maps $\mathcal{L}_2(\lambda,Z_2)$ onto $\mathcal{L}_2(-\infty,+\infty)$, which is given by the adjoint,

$$U^*(x,y) = \frac{1}{\sqrt{2\pi}} (xy)^{-1/4} (\cos x\sqrt{y}, \sin x\sqrt{y}) .$$

Next we introduce a condition on the function q which will ensure that in the $-\Delta$ representation M_q is represented by a locally gentle operator. It reads as follows:

<u>Condition $(7.1)_A$</u>

$$\int_{-\infty}^{\infty} (1+t^2)|q(t)|dt < \infty .$$

One can easily find the operator M_q in the $-\Delta$ representation, that is the operator UM_qU^*. For, at least formally,

$$(7.1) \qquad UM_qU^*(x,y) = \int_{-\infty}^{\infty} U(x,t)q(t)U^*(t,y)dt$$

and this can be also established rigorously under the present condition. Note that for fixed values of the variables x,y and t, $U(x,t)q(t)U^*(t,y)$ is a linear operator on Z_2. Moreover, we see that the matrix of this operator is given by,

$$U(x,t)q(t)U^*(t,y)$$

$$= \frac{1}{2\pi} (xy)^{-1/4} q(t) \begin{pmatrix} \cos t\sqrt{x} \cos t\sqrt{y}, & \cos t\sqrt{x} \sin t\sqrt{y} \\ \sin t\sqrt{x} \cos t\sqrt{y}, & \sin t\sqrt{x} \sin t\sqrt{y} \end{pmatrix} .$$

Insertion of this relation in (7.1) yields,

$$(7.2)\quad K(x,y) =$$

$$= \frac{1}{2\pi}(xy)^{-1/4}\begin{pmatrix} \int_{-\infty}^{\infty} \cos t/x \cos t/y \; q(t)dt, & \int_{-\infty}^{\infty} \cos t/x \sin t/y \; q(t)dt \\ \int_{-\infty}^{\infty} \sin t/x \cos t/y \; q(t)dt, & \int_{-\infty}^{\infty} \sin t/x \sin t/y \; q(t)dt \end{pmatrix}$$

after setting $K = UM_qU^*$. Next we maintain that this kernel satisfies the conditions of Theorem 6.1, that is we maintain the following:

Lemma 7.2

Suppose that the function q satisfies Condition 7.1. Then the operator K defined by (7.2) can be written as the sum of a gentle operator and an almost gentle operator of finite rank. Moreover both of these operators are gentle with exponent $\alpha > 1/2$ in any closed subinterval of $[0,\infty)$ not containing the point 0.

Let $G(x,y)$ denote the matrix factor entering equation (7.2). Then according to Condition 7.1 one can differentiate under the integral sign provided that $x \neq 0$ and $y \neq 0$. Hence this operator is gentle with exponent 1 in any closed interval not containing the point zero. Next we claim that G is also gentle with some positive exponent on the entire interval $[0,\infty)$. For, consider the operator $\tilde{G}(x,y) = G(x^2,y^2)$ and note that in view of the change of variables the kernel $\tilde{G}(x,y)$ is

differentiable including the points $x=0$ and $y=0$. Hence $\tilde{G}$ is gentle with exponent 1 on the entire interval $[0,\infty]$. From this, we conclude in view of the elementary inequality

$$|(x+h)^{1/2} - x^{1/2}| = O(1)h^{1/2} ,$$

that G is gentle with exponent 1/2 on the entire interval $[0,\infty]$. From this, in turn, we conclude that Lemma 1 of Appendix IV applies to the operator $K(x,y) = (xy)^{-1/4} G(x,y)$, which establishes the validity of the present lemma.

In other words, we have established that Theorem 6.1 applies to the operator M+K. This yields, remembering that by definition M+K is unitarily equivalent to $-\Delta + M_q$, the following:

Theorem 7.1

Suppose that the function q satisfies Condition 7.1_A. Then the operator $-\Delta + M_q$ is essentially selfadjoint. The continuous part of its strictly selfadjoint extension, $(-\Delta + M_q)_c$ is unitarily equivalent to $-\Delta$, in particular it is absolutely continuous. Moreover, in the $-\Delta$ representation $(-\Delta + M_q)_c$ admits a spectral transformation of the form $(1+\Gamma Q_1)(1+\Gamma Q_2)$ where Q_1 is gentle and Q_2 is locally gentle.

Remark

One could slightly relax Condition $(7.1)_A$. For, according to equation (7.2), the behavior of q at infinity determines the exponent of K. The behavior of q stipulated by Condition $(7.1)_A$

ensured that K was differntiable ''away from zero''. Next, instead of this, we only stipulate that

$$\int_{\infty}^{\infty} (1 + |t|^{\varepsilon})|q(t)|dt < \infty$$

for some $\varepsilon > 0$. Then one can take Holder quotients under the integral sign and this shows that K is gentle ''away from zero''. Now the expnent of K is possibl- less than 1/2 and so Theorem 6.1 is no longer applicable. Nevertheless, Theorem 6.2 applies to operator M+K. This is far from being obvious, for one has to establish Condition $(0, \infty)$. This in turn involves a straight-forward but somewhat lengthy argument. Since the theorem that we obtain is still weaker than the corresponding theorem of Titchmarsh admitting the case $\varepsilon = 0$, we skip the details.

B. Schrödinger operator in free space dimension $n \geq 2$.

It was stated by O.A. Laqyzhenskayo and L.D. Faddeev, [B3] that under suitable conditions on the function q, the operator M_q is gentle with respect to $-\Delta$ in case n=3. Later this was established by J. Schwartz, [B5], for the case of $n \geq 2$. under the following condition:

Condition $(7.1)_B$

$$\int (1+t^2) \ |D^j q(t)|dt < \infty \quad \text{for} \quad j \leq n-1 \quad ,$$

and for any j-th order partial derivative $D^j q$.

7.7

Note that originally the notion of gentleness was defined with respect to M, the multiplication operator. We say that M_q is gentle with respect to $-\Delta$, if in the $-\Delta$ representation M_q is represented by a gentle operator. That is the operator UM_qU^* is gentle with respect to M where U is a spectral transformation of $-\Delta$. Next we turn to the description of this operator.

For this purpose we need U, which is essentially the Fourier transformation. In order to describe U in more specific terms, we introduce ''polar coordinates'' in $\mathcal{L}_2(R_n)$ as follows: let $\mathfrak{N}$ denote the $\mathcal{L}_2$ -space over the n-dimensional unit sphere and let λ be the restriction of the Lebesques measure to $[0,\infty]$. Then according to the formulae connecting cartesian and polar coordinates the spaces $\mathcal{L}_2(R_n)$ and $\mathcal{L}_2(\lambda, \mathfrak{N})$ are isomorphic via the transformation $J: \mathcal{L}_2(R_n) \to \mathcal{L}_2(\lambda, \mathfrak{N})$ defined by

$$Jf(r)(\vec{x}) = 2\pi)^{-\frac{n}{2}} \quad r^{\frac{n-1}{2}} \quad f(r\,\vec{x}) \quad .$$

Here Jf is an $\mathfrak{N}$ valued function, Jf(r) denotes the value of this function at the point r, which in turn has the value $Jf(r)(\vec{x})$ at the unit vector $\vec{x}$. Evidently F, the Fourier transformation on $\mathcal{L}_2(R_n)$ carries $-\Delta$ into multiplication by the square of the independent vector variable. In other words, according to the definition of J,

$$-\Delta = F^*J^*M^2JF \ ,$$

where M is the multiplication operator on $\mathcal{L}_2(\lambda, \mathfrak{N})$. Next introduce r as new independent variable via the unitary operator T on $\mathcal{L}_2(\lambda, \mathfrak{N})$,

7.8

$$Tf(r) = \frac{1}{\sqrt{2}} r^{-1/4} f(r^{1/2}) .$$

Then clearly

$$M^2 T = TM ,$$

and thus

$$-\Delta = F^* J^* T^* MTJF ,$$

in other words

$$-\Delta = U^* MU \quad \text{with} \quad U = TJF .$$

Therefore, the operator $-\Delta + M_q$ is unitarily equivalent to $M + UM_qU^*$. Now setting $K = UM_qU^*$, from the formula $U = TJF$ we infer that K is an integral-operator and the values of its kernel are in turn integral-operators on the accessory space $\mathfrak{H}$. More specifically we infer that

$$(7.3)\quad K(r,s)(\vec{x},\vec{y}) = (2\pi)^{-n}(rs)^{\frac{n-2}{4}} \int e^{i\sqrt{r}\,(\vec{x},\vec{y})} e^{-i\sqrt{s}\,(\vec{y},\vec{t})}\, q(\vec{t})\, dt .$$

Thus we have found the operator M_q in the $-\Delta$ representation. Using this operator one can state the gentleness of M_q with respect to $-\Delta$ as follows:

Lemma. (J. Schwartz, [B5])

Suppose that the function q satisfies Condition $(7.1)_B$. Then for $n \geq 2$ in the $-\Delta$ representation, the operator M_q is represented by a gentle operator. Moreover, this operator, that is the operator K appealing in (7.3) is a completely gentle operator in $\mathfrak{G}'(\alpha,\beta,\lambda,\mathfrak{H})$. Here λ is the restriction of the Lebesgue measure to $[0,\infty]$ and $\mathfrak{H}$ is the $\mathfrak{L}_2$-space over the n-dimensional unit sphere.

The lemma makes no explicit statement on the exponent α, nevertheless its proof, [B5], shows that $\alpha \geq 1/2$ on the interval $[0,\infty]$ and $\alpha = 1$ on any closed subinterval of $[0,\infty]$ not containing the point 0. Therefore, Theorem 6.1 applies to the operator M+K and remembering that by definition M+K is unitarily equivalent to $-\Delta+M_q$ it yields the following theorem, which is essentially a special case of a result of Povzner and Ikebe.

Theorem 7.2

Suppose that the function q satisfies Condition 7.2. Then the operator $-\Delta+M_q$ is essentially selfadjoint. The continuous part of its unique strictly selfadjoint extension $(-\Delta + M_q)_c$, is unitarily equivalent to $-\Delta$, in particular it is absolutely continuous. Moreover, in the $-\Delta$ representation $(-\Delta + M_q)_c$ admits a spectral transformation of the form $(1 + \Gamma Q_1)(1 + \Gamma Q_2)$ where Q_1 is gentle and Q_2 is locally gentle.

Note that in spite of the highly restrictive Condition 7.2 the essential selfadjointness of $-\Delta + M_q$ is not evident. This has been established under more general condition by T. Kato, [D1].

C. Example of a differential operator with boundary conditions.

We ask the question, how does a change in the boundary conditions affect the spectrum of a differential operator? First one is tempted to say that such changes induce ''gentle'' changes in

the spectrum. This, however, is far from being the case, for N. Aronszajn, [C2], gave an example of a Sturm-Liouvielle operator with the following property: for one boundary condition it has a ''pure point spectrum'', while for other boundary conditions it has a ''pure continuous spectrum''. Nevertheless, one feels that such a phenomenon is a pathology although one does not know why. The author conjectures that ''in general'' a change in the boundary conditions ''amounts'' to a locally gentle perturbation and thus leaves the continuous spectrum invariant. Next let us formulate this conjecture in more specific terms and establish its validity for the following, very special differential operator: The operator D_α is the closure of

$$D_\alpha f(x) = -f''(x)$$

defined for those twice continuously differentiable functions on $[0,\infty]$, which have compact support and satisfy the boundary condition

$$\cos \alpha f(0) + \sin \alpha \, f'(0) = 0 \quad .$$

Strictly speaking, the operator D_α is not a perturbation of the operator D_0, for they do not have a common domain. Nevertheless it has been pointed out by F. Wolf, [C3], that this leads to a perturbation problem for the corresponding resolvents, in particular for the operators $(1+D_\alpha)^{-1}$ and $(1+D_0)^{-1}$. Next we maintain that this perturbation is locally gentle, more specifically we maintain the following:

7.11

Lemma

The operator $(1+D_\alpha)^{-1} - (1+D_o)^{-1}$ is of rank one, moreover in the $(1+D_o)^{-1}$ representation it is represented by a locally gentle operator in $\mathcal{G}(\alpha/(01))$ with $\alpha > 1/2$.

In order to establish the lemma, recall from the theory of Sturm-Liouvielle operators that $(1+D_\alpha)^{-1}$ and $(1+D_o)^{-1}$ are integral operators with kernel:

$$(7.4)_\alpha \qquad (1+D_\alpha)^{-1}(x,y) = \begin{cases} \phi_\alpha(x)\psi(y) & = \quad x \leq y \\ \psi(x)\phi_o(y) & = \quad x \geq y \end{cases}$$

$$(1+D_o)^{-1}(x,y) = \begin{cases} \phi_o(x)\psi(y) & = \quad x \leq y \\ \psi(x)\phi_o(y) & = \quad x \geq y \end{cases}$$

Here the functions ϕ_o, ϕ_α, ψ satisfy the equation $-f''+f = 0$, ϕ_o and ϕ_α satisfy the corresponding boundary conditions and ψ is square inegrable ''near infinity''. Moreover

$$W(\phi_o, \psi) = W(\phi_\alpha, \psi = 1,$$

where W denotes the Wionskian. Now subtraction of $(7.4)_\alpha$ from $(7.4)_o$ yields

$$[(1+D_\alpha)^{-1} - (1+D_o)^{-1}](x,y) = \begin{cases} (\phi_\alpha(x)-\phi_o(x))\psi(y) & x \leq y \\ \psi(x)\phi_\alpha(y)-d_o(y) & x \geq y \end{cases} .$$

Next observe that the function $\phi_\alpha-\phi_o$ is a constant multiple of the function ψ, for $W(\phi_\alpha-\phi_o,\psi) = 0$. Therefore, if c denotes this constant,

we have

$$(1+D_c)^{-1} - (1+D_o)^{-1} = c\psi \; x \; \psi.$$

Thus the fact that this operator is of rank one has been established.

In order to show that in the $(1+D_o)^{-1}$ representation this operator is represented by a locally gentle operator in $\mathcal{G}(\alpha|(0,1))$, we need the following version of the spectral mapping theorem:

Lemma.

Let A be an operator with simple absolutely continuous spectrum and let f be a function which is monotone on $\sigma(A)$, the spectrum of f(A) and continuously differentiable with the exception of a finite number of points. Then the spectrum of f(A) is simple, absoloutely continuous, given by the set A. Moreover if A admits a spectral transformation, U_A, then so does f(A) and

$$U_{f(A)}(x,y) = U_A(f^{-1}(x),y)\,\rho(x)$$

$$\rho(x) = \sqrt{\left|\frac{d}{dx} f^{-1}(x)\right|} \quad .$$

As is well known the spectral transformation of $\sqrt{D_o}$ is given by the kernel $\frac{1}{\sqrt{2\pi}}\sin xy$. From this fact and from the lemma we conclude that, U, the spectral transformation of $(1+D_o)^{-1}$ is given by

$$U(x,y) = \frac{1}{\sqrt{2\pi}}\, x^{-1}\left(\frac{1}{x} - 1\right)^{-1/4} \sin\sqrt{\frac{1}{x} - 1}\; y$$

hence

$$U\psi(x) = \frac{1}{\sqrt{2\pi}}\, x^{-1}\left(\frac{1}{x} - 1\right)^{-1/4} \int_0^\infty \sin\sqrt{\frac{1}{x} - y}\;\psi(y)\,dy.$$

From this formula in turn, and from definition of ψ as a solution of $-\psi'' + \psi = 0$ which is square integrable at infinity, we conclude that the function $U\psi$ is differentiable in the open interval $(0, 1)$. Hence, in particular, it is locally gentle with exponent $\alpha > 1/2$. This completes the proof of the Lemma.

Now from the local gentleness of the function $U\psi$ we can easily derive that the continuous spectrum of D_α is absolutely continuous and of multiplicity one. For, let M be the multiplication operator on $\mathcal{L}_2(01)$. Then from the fact that $U\psi$ is locally gentle with exponent $\alpha > 1/2$ and from Theorem 5.3 we conclude that the continuous part of $M + cU\psi \otimes U\psi$ is unitarily equivalent to M. On the other hand according to the definition of U, we have

$$\begin{aligned} M + cU\psi \times U\psi &= U(1+D_o)^{-1}U^* + U[(1+D_\alpha)^{-1} - (1+D_o)^{-1}]U \\ &= U(1+D_\alpha)^{-1}U \quad . \end{aligned}$$

Thus $(1+D_\alpha)^{-1}$ is unitarily equivalent to $M + cU\psi \otimes U\psi$. Therefore, the continuous part of $(1+D_\alpha)^{-1}$ is unitarily equivalent to the continuous part of this operator and hence to M. Hence the continuous part of $(1+D_\alpha)^{-1}$ is absolutely continuous and its multiplicity is one. From this in turn we conclude via the spectral mapping theorem that this is also true about the operator D_α.

7.14

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Appendix I

Proof of the local gentleness propositions in $\mathfrak{G}(\alpha|[\delta_1,\delta_2],\lambda,\mathfrak{H})$.

First recall from Section 2 that the space $\mathfrak{G}_{aux}$ was defined as the set of those bilinear forms of finite rank which can be written as $\sum h_i >< g_i$, where the forms $\{h_i\}$, $\{g_i\}$ can be identified with measuable functions. Then the space $\mathfrak{G} = \mathfrak{G}(\alpha|[\delta_1,\delta_2],\lambda,\mathfrak{H})$ was defined to be the set of such forms in $\mathfrak{G}_{aux}$ to which there is an interval $[\tilde{\delta}_1,\tilde{\delta}_2] \supset [\delta_1,\delta_2]$ such that the functions identified with the form are α - Holder continuous in $[\tilde{\delta}_1,\tilde{\delta}_2]$. From this definition and from the definition of the Γ - transformation on this space, that is from equation (2.1) it is clear that the first three statements of Section 2 on local gentleness hold. Therefore, we start this Appendix by establishing the fourth statement.

4. For every G in $\mathfrak{G}'$ the product of the operator C_δ and of the form ΓG, $C_\delta \Gamma G$, is bounded.

According to our simplifying assumption a locally gentle form is a finite linear combination of dyads. Therefore, it suffices to establish the above statement for a single dyad only. In doing this, we shall find it convenient to denote by $[g]$ the multiplication operator assigned to the measurable function g, that is the operator $\mathcal{L}_2(\lambda,\mathfrak{H})$ defined by $[g]f(x) = (g(x),f(x))$. Then using this notation, according to the definition of the Γ - transformation, that is according to (2.1) we have

I-2

$$\langle C_\delta f_1, \Gamma(g_1 >< g_2) f_2 \rangle = \langle f_1, C_\delta[\bar{g}_1]\, H_+[\bar{g}_2] f_2 \rangle,$$

in short

(I-1) $$C_\delta \Gamma(g_1 >< g_2)\ [C_\delta g_1] H_+[g_2]\ ,$$

where H_+ is the augmented Hilbert transformation. Thus the form $C_\delta \Gamma(g_1 >< g_2)$ is bounded if the operator appearing on the right side of (I-1) is bounded. Next we maintain that this is the case for functions g_1, g_2, which are gentle over $[\delta_1, \delta_2]$. For, for such functions the function $C_\delta \bar{g}_1$ is bounded, hence so is the operator $C_\delta[\bar{g}_1]$. Next, one is tempted to conclude the boundedness of the operator $H_+[g_2]$ from the fact that H_+, the augmented Hilbert transformation is bounded. However, in view of the fact that $[g_2]$ is a possibly unbounded operator, we have to use an additional feature of the Hilbert transformation. By assumption g_2 is gentle over $[\delta_1, \delta_2]$, hence there is an interval $[\tilde{\delta}_1, \tilde{\delta}_2] \supset [\delta_1, \delta_2]$ such that g_2 is Holder continuous in the extended interval. Hence the function $C_{\tilde{\delta}} g_2$ is bounded and therefore

(I-2) $$\| [C_\delta g_1] H_+ [C_{\tilde{\delta}} g_2] \| < \infty\ .$$

On the other hand, we have the following elementary inequality;

$$\left| c(x) \int \frac{(1-\tilde{c}(y))(g_2(y), f(y))}{x-y}\, d\lambda(y) \right|^2 \leq \sup_{x \neq y} \left| \frac{c(x)(1-\tilde{c}(y))}{x-y} \right|^2 \cdot$$

$$\cdot \int (g_2(y), g_2(y))\, d\lambda(y) \cdot \int (f(y), f(y))\, d\lambda(y)\ ,$$

where c resp. $\tilde{c}$ denote the characteristic function of the interval

$[\delta_1,\delta_2]$ resp. $[\breve{\delta}_1,\breve{\delta}_2]$. Now by definition $[\delta_1,\delta_2]$ is a proper closed subinterval of $[\tilde{\delta}_1,\tilde{\delta}_2]$ and from this fact we conclude that the ''sup'' - term in the above inequality is finite. Therefore

$$||C_\delta H_+(1-C_{\tilde{\delta}})g_2 f|| = O(1)||f|| \quad ,$$

holds for every function f in $\mathcal{L}_2(\lambda,\breve{\mathfrak{H}})$ where the constant O(1) is independent of f. In other words

$$||C_\delta H_+[(1-C_{\tilde{\delta}})g_2]|| < \infty, \tag{I-3}$$

from which in turn, in view of (I-2) and (I-1) we conclude the boundedness of $C_\delta\Gamma(g_1 >\!< g_2)$.

Next we turn to the proof of the local gentleness propositions.

<u>Proposition 1</u>

<u>To every G in $\mathcal{E}_{aux}$ there is a dense set $\mathfrak{D}$ such that</u>

$$M\Gamma G - \Gamma G M = G \text{ on } \mathfrak{D}. \tag{I-4}$$

Proof.

As noted previously it is no loss of generality to assume that $G = g_1 >\!< g_2$. Since the set of functions with bounded support is dense in $\mathcal{L}_2(\lambda,\tilde{\mathfrak{H}})$ we may also assume that the support of the measure λ, S_λ, is bounded. Now let $\{S_n\}$ be an increasing family of subsets of S_λ, such that the functions g, and g_2 are bounded on each of the set S_n, moreover these sets approximate S_λ in measure, that is

I-4

$$\lim_{n\to\infty} \lambda(S_n) = \lambda(S_\lambda)$$

Next set

$$\mathcal{D} = \bigcup_n \mathcal{L}_2(\lambda, S_n)$$

Then clearly the set $\mathcal{D}$ is dense in $\mathcal{L}_2(\lambda, \mathcal{T})$ and it is not hard to verify that relation (I-4) holds on this set. For, from the definition of the form $\Gamma_\varepsilon G$, that is from relation (2.1), we have

$$\langle f_1, M\Gamma_\varepsilon G f_2\rangle - \langle f_1, \Gamma_\varepsilon G M f_2\rangle = \langle f_1, G f_2\rangle - i\varepsilon \langle f_1, \Gamma_\varepsilon G f_2\rangle.$$

Upon insertion of relation (I-1) in this equation we obtain

$$\langle [g_1] M f_1, H_\varepsilon [g_2] f_2\rangle - \langle [g_1] f_1, H_\varepsilon M [g_2] f_2\rangle =$$

I-5

$$= \langle f_1, G f_2\rangle - i\varepsilon \langle [g_1] f_1, H_\varepsilon [g_2] f_2\rangle .$$

Now observe that according to the construction of the sets $\{S_n\}$ the operator $[g_1]$, and $[g_2]$ are <u>bounded on</u> $\mathcal{L}_2(\lambda, S_n)$. Observe also that according to construction each of the sets S_n is bounded, hence so is the operator M on $\mathcal{L}_2(\lambda, S_n)$. Therefore the bilinear forms in (I-5) admit a limit, moreover the limit can be obtained by setting $\varepsilon = 0$. This in turn, in view of (I-1) establishes Proposition 1.

<u>Proposition 2.</u>

<u>For every G in $\mathcal{G}$ the forms $C_\delta \Gamma G \cdot G^* C_\delta$ and $C_\delta G \cdot \Gamma G^* C_\delta$ are in $\mathcal{G}_{aux}$.</u>

Again it suffices to establish the proposition for dyads only, and accordingly let $G = g_1 \rangle\langle g_2$. Then

$$C_\delta G = C_\delta g_1 \rangle\langle g_2 \text{ and } G^* C_\delta = g_2 \rangle\langle C_\delta g_1 .$$

On the other hand, according to (I-1), the local gentleness of G implies that the form $C_\delta \Gamma G$ is bounded and $C_\delta \Gamma G = [C_\delta g_1]H_+[g_2]$. Now from the above relations we conclude that

$$C_\delta \Gamma G \cdot G^* C_\delta = [C_\delta g_1]H_+[g_2]\cdot(g_2 >< C_\delta g_1) =$$
$$=[C_\delta g_1]H_+ g_2^2 >< C_\delta g_1 \ .$$

Finally from this relation we conclude the validity of the proposition. For, by assumption $g_1 >< g_2$ is in $\mathcal{G}$, in particular it is in $\mathcal{G}_{aux}$, hence g_2^2 is integrable and . therefore, its Hilbert transform is measurable. On the other hand, by assumption $[\delta_1, \delta_2]$ is bounded and g_1 is gentle over this interval, hence $C_\delta g_1$ is square integrable.

Proposition 3.

For every G in ,

$$C_\delta \Gamma G \cdot \Gamma G^* C_\delta = \Gamma(C_\delta G \Gamma G^* C_\delta + C_\delta \Gamma G \cdot G^* C_\delta) \ .$$

Proof. Following Friedrichs', [B1], we start from the identity

$$\frac{1}{x-y+i\varepsilon} \cdot \frac{1}{x-z+i\varepsilon} = \frac{1}{x-z+2i\varepsilon}\left(\frac{1}{x-y+i\varepsilon} + \frac{1}{y-z+i\varepsilon}\right),$$

from which we infer

$$C\Gamma_\varepsilon G\cdot\Gamma_\varepsilon G^* C = \Gamma_{2\varepsilon}(CG\cdot\Gamma_\varepsilon G^* C + C\Gamma G\cdot G^* C). \qquad \text{(I-6)}$$

Here and in the following, for convenience of notation, we omitted the subscript δ appearing in C. Now this relation makes the proposition reasonable, for the proposition states that after setting $\varepsilon = 0$ in this relation we obtain two forms which are equal

on a dense set.

First we shall show that there is a dense set $\mathcal{D}$ in $\mathcal{L}_2(\lambda,\tilde{\mathfrak{H}})$ such that on this set the family of forms appearing on the right side admits a limit as $\varepsilon \to +0$, moreover this limit is obtained by setting $\varepsilon = 0$. We shall show this by establishing the following slightly more general statement: <u>there is a dense set $\mathcal{D}$ such that on $\mathcal{D}$ the double limit</u>

(I-7) $\lim\limits_{\eta \to 0} \; \lim\limits_{\varepsilon \to 0} \quad \Gamma_\eta(CG\cdot\Gamma_\varepsilon G^*C + C\Gamma G\cdot G^*C)$

<u>exists, moreover it is equal to</u>

(I-8) $\Gamma(CG\cdot\Gamma G^*C + C\Gamma G\cdot G^*C)$.

It is well known, [ ], that the existence of the double limit is ensured by the existence of the iterated limits provided that one of the iterated limits is uniform. Therefore, according to the definition of the Γ transformation, the statement concerning the forms in (I-7) and (I-8) is implied by the following:

<u>Lemma</u>.

<u>Suppose that $G = g_1 \times g_2$ is in $\mathcal{G}_{aux}$ that is g_1 is measurable and g_2 is square integrable. Then there is a set $\mathcal{D}$ which is dense in $\mathcal{L}_2(\lambda,\tilde{\mathfrak{H}})$ and is such that</u>

(I-9) $\lim \quad \langle f, \Gamma_\eta(C\Gamma_\varepsilon G^*)f \rangle = \langle f, \Gamma_\eta(G\Gamma G^*)f \rangle, \; f \in \mathcal{D},$

<u>moreover this limit is uniform in η.</u>

Since the set of functions with bounded support is dense in

$\mathcal{L}_2(\lambda, \int\!\!\!\!\!\!\!\!\int\!\!\!\int)$ we may assume that the support of λ, S_λ, is bounded. On the other hand, in analogy with relation (I-1) we have

$$\Gamma_\eta(G\Gamma_\varepsilon G^*) = \Gamma_\eta(g_1 >< g_2\Gamma_\varepsilon G^*) = -[g_1]H_\eta[g_1 H_\varepsilon g_2^2]. \tag{I-10}$$

Next we shall construct a family of sets $\{S_n\}$ such that on each of the sets S_n the following three statements hold:

a) The functions g_1 and g_2 are bounded,

b) The family of functions $g_1 H_\varepsilon g_2^2$ converges uniformly as $\varepsilon \to +0$,

c) The family of sets $\{S_n\}$ approximates S_λ in measure, that is

$$\lim \lambda(s_n) = \lambda(S_\lambda) \quad .$$

In order to satisfy condition a, consider level sets of the functions g_1 and g_2, that is the sets,

$$S_{n,a} = E[x: |g_1(x)| \leq n, \quad |g_2(x)| \leq n].$$

In order to satisfy condition b, note that by assumption g_2^2 is integrable. Hence according to the $\mathcal{L}_1$-version of the Plemelj-Privalov theorem, [E3], the family of functions $H_\varepsilon g_2^2$ converges pointwise, almost everywhere. From this we conclude convergence in measure, from which in turn, using Egorov's theorem, [E2,b], we conclude uniform convergence on the complement of an exceptional set of measure, say 1/n. Let $S_{n,b}$ denote this set. Then clearly the family of functions $g_1 H_\varepsilon g_2^2$ converges uniformly on $S_{n,a} \cap S_{n,b}$. Finally to ensure condition c, we form a monotone sequence from these sets, by defining

I-8

$$S_n = \bigcup_{k=1}^{n} S_{k,a} \cap S_{k,b} \quad .$$

Then, clearly this family of sets satisfies the conditions a,b, and c. Finally we set

$$\mathcal{D} = \bigcup \mathcal{L}_2(\lambda/S_n, \mathfrak{H})$$

and maintain that this set satisfies the conclusion of the lemma. For, from statement c, we see that $\mathcal{D}$ is dense. Then from statement b, we infer the validity of relation (I-9). For, insertion of relation (I-10) in (I-9) yields

I-11 $\langle f_1, \Gamma_\eta (G \Gamma_\varepsilon G^*) f \rangle = - ([g_1] f_1, H_\eta [g_1 H_\varepsilon g_2^2] f) \quad .$

Note that the right side of this equation is an $\mathcal{L}_2(\lambda, \mathfrak{H})$ inner product. Also note that according to statement b, the family of functions $g_1 H_\varepsilon g_2^2$ converge uniformly on S_n, thus the corresponding family of operators $[g_1 H_\varepsilon g_2^2]$ converges in the operator norm on $\mathcal{L}_2(\lambda/S_n, \mathfrak{H})$. From this fact, in view of the uniform boundedness of the family of operators H_η, we conclude that the expression in (I-11) admits a limit which is uniform in η. This establishes the lemma.

Now, from the lemma we can easily derive the proposition. For, as noted earlier, the lemma implies that the family of forms entering the ight side of (I-6) converges on $\mathcal{D}$, moreover the limit is obtained by setting $\varepsilon = 0$. On the other hand, from the local gentleness of G we conclude that the family of operators

which can be identified with the family of forms $C\Gamma_{\varepsilon}G$ converges strongly to $C\Gamma G$. From this fact and from the boundedness of the operator $C\Gamma G$, we conclude that the family of operators $C\Gamma_{\varepsilon}G\cdot\Gamma_{\varepsilon}C^{*}C$ converges strongly to the operator $C\Gamma G\cdot\Gamma G^{*}C$. Therefore, the family of forms $C\Gamma_{\varepsilon}G\cdot\Gamma_{\varepsilon}G^{*}C$ converges to the form defined by the limit operator. This establishes the proposition.

Remark

The key fact in the proof of the proposition is stated in the lemma, which refers to forms in $\mathcal{E}'_{aux}$. The local gentleness of G was used inasmuch as we used the fact that the operator $C\Gamma_{\varepsilon}G$ converges strongly to the operator $C\Gamma G$. Now in case G is non locally gentle, it may happen that one has strong convergence on some subspace of $\mathcal{L}_2(\lambda, \mathfrak{H})$. In this case, the product identity holds on the corresponding subspace. We shall make use of this fact in the next Appendix.

Appendix II

Arbitrary, selfadjoint perturbations of finite rank.

According to a theorem of T. Kato, [A3], under a trace class perturbation the absolutely continuous parts of the perturbed and unperturbed operators are unitarily equivalent. He also showed that a unitary transformation establishing this equivalence can be obtained as the limit of ''wave operators''. Later L. de Branges gave another construction, [A5], of such a unitary transformation, using his theory of ''Hilbert spaces of entire functions''. Thus under a trace class perturbation spectral transformation for the absolutely continuous part of the disturbed operator are well known.

The content of this appendix is the observation that <u>for arbitrary selfadjoint perturbations of finite rank the bilinear form U_δ introduced in (5.12) is bounded on the absolutely continuous subspace of M+K. Furthermore the transformation identified with it is a spectral transformation of the absolutely continuous part of M+K over the interval</u> $[\delta_1, \delta_2]$. We shall establish this fact analogously to the way we established that under the conditions of Section 5, the form U_δ could be identified with the spectral transformation of the entire operator M+K.

We start with the following version of the resolvent loop integral formula.

II-2

Lemma 1.

Let A be an arbitrary strictly selfadjoint operator with resolvent $R(z)$. Then there exists a countable set of vectors $\{f_j\}$ which is dense in the absolutely continuous subspace of A and is such that for an arbitrary fixed vector f_j and measurable set S,

$$(II.1)\quad (f_j, E(S)f_j) = \lim_{\varepsilon\to+0} \int_S (f_j,(R(x+i\varepsilon) - R(x-i\varepsilon))f_j)dx \quad .$$

We shall show that this relation is a consequence of the countable additivity of the spectral projectors and of the following, simpler statement:

Lemma 2.

Let A be the operator appearing in the previous lemma which acts on the abstract Hilbert space $\mathfrak{H}$. Then there exists a countable set of vectors $\{f_j\}$ which is dense in the absolutely continuous subspace of A and is such that for any fixed vector f_j, the family of numerical valued functions

$$(II.2)\qquad (f_j, (R(x+i\varepsilon) - R(x-i\varepsilon))f_j)$$

admits a bound in $x\varepsilon[-\infty, +\infty]$ which is uniform in ε.

Proof. Let E_y denote the spectral projector of A over the interval $(-\infty,y)$. Then we maintain that there is a countable set of vectors, $\{f_j\}$, which is dense in the absolutely continuous subspace of $A\mathfrak{H}_{abs}$. and is such that for fixed $\{f_j\}$ the function

$$(II.3)\qquad \frac{d(f_j,E_y f_j)}{dy}$$

is bounded.

II-3

For, let V be a spectral transformation of the absolutely continuous part of A. That is V is a unitary transformation which maps $\mathfrak{H}_{abs}$, onto some $\mathcal{L}_2(\lambda, \{\mathfrak{N}\})$ space in such a way that

$$A = V^*MV \quad \text{on} \quad \mathfrak{H}_{abs} ,$$

where M is the multiplication operator on $\mathcal{L}_2(\lambda, \{\mathfrak{N}\})$. Now let $\{v_j\}$ be an arbitrary dense family of bounded continuous functions in $\mathcal{L}_2(\lambda, \{\mathfrak{N}\})$. Then we claim that the family of vectors V^*v_j in $\mathfrak{H}$, which is clearly dense in $\mathfrak{H}_{abs}$ satisfies relation (II-3). For, since V is a spectral transformation of A, we have

$$VE_yV^* = C_y ,$$

where C_y is the operator of mult. by the characteristic function of $(-\infty, y)$. Hence

$$\frac{d}{dy} (V^*v_j, E_yV^*v_j) = \frac{d}{dy} (v_j, VE_yV^*v_j) =$$

$$= \frac{d}{dy} (v_j, C_yv_j) = \frac{d}{dy} \int_{-\infty}^{y} (v_j(x), v_j(x))d\lambda(x) \leq (v_j(y), v_j(y)) .$$

Thus the vectors V^*v_j satisfy relation (II-3).

Next we maintain that the family of functions the vectors satisfy

$$(f_j, (R(x+i\varepsilon) - R(x-i\varepsilon))f_j)$$

remains bounded uniformly in ε. For, according to the theorem on spectral resolution

$$R(x+i\varepsilon) = \int \frac{1}{x+i\varepsilon-y} \, dE(y) ,$$

hence

$$(f_i, (R(x+i\varepsilon) - R(x-i\varepsilon))f_j) = 2i \int_{-\infty}^{\infty} \frac{\varepsilon}{(x-y)^2+\varepsilon^2} \cdot \frac{d(f_j, E(y)f_j)}{dy} \, dy .$$

Finally inserting the definition of the vector f_j, that is (II-3), in this relation, and considering the fact that

$$\int_{-\infty}^{\infty} \frac{\varepsilon}{(x-y)^2+\varepsilon^2}\, dy = \int_{-\infty}^{\infty} \frac{1}{t^2+1}\, dt < \infty \quad ,$$

we obtain the validity of the lemma.

Next we turn to the proof of the original lemma. First, note that in view of the assumption $f_j \in {}_{abs}$, for fixed f_j, the set function appearing on the left side of (II-1) is continuous with respect to the measure λ. On the other hand, from Lemma 2 we conclude that this is also true about the right side. From these facts, in turn, and from the fact that a measurable set can be approximated from above by an open set and from below by a closed, we conclude that it suffices to establish (II-1) for open and closed sets only. Finally, in view of the fact that in case S is an interval formula (II-1) yields the original version of the loop integral formula, it suffices to establish it for open sets only.

Accordingly, let S be an open set and write it as a countable union of disjoint open intervals, say $S = \cup \tilde{S}_n$. Next set

$$S_n = \cup \tilde{S}_k,$$

where the union is taken over those intervals whose length is at least $1/n$. Now each of the sets S_n is the finite union of intervals and, therefore, the resolvent loop integral formula [E2,c] yields,

$$(\text{II-4})\qquad (f_j,E(S_n)f_j) = \lim_{\varepsilon\to+0} \frac{1}{2\pi i}\int_{S_n}(f_j,(R(x+i\varepsilon)-R(x-i\varepsilon))f_j)dx\ .$$

On the other hand, the countable additivity of the Lebesgue measure yields

$$\lambda(S) = \lim_n \lambda(S_n)\ ,$$

which in turn, in view of $f_j\ \varepsilon\ \mathfrak{H}_{abs}$, yields

$$(f_j,E(S)f_j) = \lim_n (f_j,E(S_n)f_j)\ .$$

Upon inserting (II-4) in this relation, we obtain;

$$(\text{II-5})\qquad \begin{aligned}(f_j,E(S)f_j) &= \lim_n (f_j,E(S_n)f_j) = \\ &= \lim_n \lim_\varepsilon \frac{1}{2\pi i}\int_{S_n}(f_j,R(x+i\varepsilon)-R(x-i\varepsilon)f_j)dx\ .\end{aligned}$$

Now clearly the iterated limits of the expression on the right side exist, moreover we conclude from Lemma 2 that the n-limit is uniform in ε. Therefore, the order of limits in (II-5) can be interchanged, hence

$$(f_j,E(S)f_j) = \lim_{\varepsilon\to+0} \frac{1}{2\pi i}\int(f_j,R(x+i\varepsilon)-R(x-i\varepsilon))f_j)dx\ ,$$

that is relation (II-1) has been established.

Next we return to our specific operator and shall establish an analogue of Lemma 5.2 which stated that under the conditions of Section 5, on a dense set of functions the perturbed resolvent could be continued weakly on the interval $[\delta_1,\delta_2]$. For arbitrary

selfadjoint perturbations of finite rank this is no longer true, all that one can claim is that the measure of the ''exceptional set'' is arbitrarily small. More specifically we have the following:

Lemma II-3

Let $[\delta_1, \delta_2]$ be a bounded interval, K an arbitrary selfadjoint operator of finite rank and let M be the multiplication operator on $\mathcal{L}_2(\lambda, \mathfrak{H})$. Then to every countable set of functions $\{f_j\}$ there is a family of subsets of $[\delta_1, \delta_2]$, $\{S_n\}$, which approximate the interval in measure and are such that for fixed f_j, the family of functions,

$$(f_j, (R(x+i\varepsilon) - R(x-i\varepsilon))f_j)$$

converges uniformly on each of the sets S_n as $\varepsilon \to +0$.

Proof. According to Lemma 5.1,

$$(f_j, R(x+i\varepsilon)f_j) = (f_j, R_o(x+i\varepsilon)f_j) +$$

$$+ \sum A^{-1}{}_{l,m}(x+i\varepsilon)(f_j, R_o(x+i\varepsilon)Kk_l)(R_o^*(x+i\varepsilon)Kk_m, f_j). \quad \text{(II-6)}$$

Then from the definition of the perturbation matrix $A_{,m}(x+i\varepsilon)$, that is from equation (5.1) we see that the matrix elements are Hilbert transforms of integrable functions. Hence the matrix elements converge almost everywhere as $\varepsilon \to +0$. From this fact, from the fact that according to S.T. Kuroda, the limit matrix is almost everywhere invertible, [A8], and from Egorov's theorem. we conclude the existence of a set $S_{n,k}$, such that:

$$\lambda([\delta_1, \delta_2] - S_{n,k}) < \frac{1}{n} ,$$

II-7

moreover the functions $A_{1,m}^{-1}$ $(x+i\varepsilon)$ converge uniformly on this set. Similarly, using again that R_o is the resolvent of the multiplication operator, we see that the factors entering (II-6) are Hilbert transforms of integrable functions. From this we conclude as above, the existence of a set $S_{n,j}$, such that:

$$\lambda([\delta_1,\delta_2] - S_{n,j}) < \frac{1}{n} \; 2^{-j}$$

moreover the other two terms under the $\sum$ - sign in (II-6) converge uniformly on this set. Finally, let

$$S_n = S_{n,k} \quad \bigcap_{j=1}^{\infty} S_{n,j} \qquad n = 1,2,\ldots \quad .$$

Then clearly on each of these sets the left side of (II-6) converges uniformly and these sets approximate the interval $[\delta_1,\delta_2]$ in measure, that is

$$\lim \; \lambda \; (S_n) = \lambda[\delta_1,\delta_2].$$

This establishes the lemma.

Finally we shall establish the statement concerning the spectral transformation of the absolutely continuous part of M+K. First, however, let us recall from Section 3 that for arbitrary selfadjoint perturbations of finite rank the Friedrichs' equation admits a solution Q in $\mathfrak{G}_{aux}$. Thus ΓQ is a densely defined form. Then we recall that the computations of Section 5, which showed that under appropriate conditions,

$$(5.11)f \qquad (f,E_\delta f) = (f, \, (1+\Gamma Q)^* C_\delta(1+\Gamma Q)f)$$

were based on the following two facts: a) under the conditions

of Section 5, the family of functions on the right side of (II-6) converge uniformly on the interval $[\delta_1, \delta_2]$, b) the resolvent loop integral formula was applicable to the interval $[\delta_1, \delta_2]$.

Now in the present case, that is in case of an arbitrary selfadjoint perturbation of finite rank, these statements do not necessarily hold on the entire interval $[\delta_1, \delta_2]$. Nevertheless, in view of lemmas 1 and 3 they do hold on any of the sets S_n entering Lemma 3. From this fact, and from the fact that the sets $\{S_n\}$ approximate the interval in measure, we conclude

(II-7) $$(f_j, E_\delta f_j) = (f_j, (1+\Gamma Q)^* C_\delta (1+\Gamma Q) f_j),$$

where $\{f_j\}$ is the set of functions entering Lemma 1. From this relation, in turn, and from the fact that the set of functions $\{f_j\}$ is dense in the absolutely continuous subspace of M+K, we conclude,

$$E_\delta P = (1+\Gamma Q)^* M (1+\Gamma Q),$$

in the same manner as equation (5.13) was derived from equation (5.11). Finally, recall that in Lemma 5.4, we described the final set of the transformation $1+\Gamma Q$ and that the product identity was the key fact in the proof. Now according to the remark of the previous Appendix, the product identity remains true on a subspace for possibly non locally gentle operators provided that the form ΓQ remains bounded on this subspace. In view of (II-7) this is the case for $1+\Gamma Q$ on the absolutely continuous subspace

of M+K. These facts show that under the present conditions the final set of $1+\Gamma Q$ is also $\mathfrak{L}_2(\lambda,\tilde{\mathfrak{h}})$. Therefore, summarizing these statements we obtain:

<u>Let K be an arbitrary selfadjoint operator of finite rank and let Q in $\mathfrak{G}_{aux}$ be the solution of the Friedrichs' equation defined by (3.3) and (3.6). Then the form $1+\Gamma Q$ is bounded on the absolutely continuous subspace of M+K, moreover the transformation identified with it is a unitary transformation which carries this part of M+K into the operator of M.</u>

Appendix III

A stronger version of the basic Lemma 3.1.

In this appendix we shall show that a necessary and sufficient condition for the point x_o in $[\delta_1,\delta_2]$ to be a pointeigenvalue of M+K on $\mathcal{B}(\alpha|[\delta_1,\delta_2],\alpha-1,\lambda,\mathfrak{H})$ is that det $A(x_o) = 0$.

The sufficiency of this condition, is the statement of the original formulation of Lemma 3.1. The necessity of this condition can be verified by a straightforward algebra that we are going to do: accordingly assume that

$$(M+K)g = x_o g \quad \text{and} \quad g \in \mathcal{B}((\alpha|[\delta_1,\delta_2],\alpha-1,\lambda,\mathfrak{H}) , \tag{III-1}$$

from which we infer

$$g(x) = \frac{\lambda_1 k_1(x)(k_1,g) + \ldots + \lambda_n k_n(x)(k_n,g)}{x_o - x} , \tag{III-2}$$

where the inner products refer to $\mathcal{L}_2(\lambda,\mathfrak{H})$. Next, taking the inner product of the function g with each of the functions k_i yields the following system of linear equations:

$$(1+\lambda_1 \int \frac{(k_1(x),k_1(x))}{x-x_o} d\lambda(x)(k_1,g)$$

$$+ \ldots + (\lambda_n \int \frac{(k_1(x),k_n(x))}{x-x_o} d\lambda(x))(k_n,g) = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$(\lambda_1 \int \frac{(k_1(x),k_n(x))}{x-x_o} d\lambda(x))(k_1,g)$$

$$+ \ldots + (1+\lambda_n \int \frac{(k_n(x),k_n(x))}{x-x_o} d\lambda(x))(k_n,g) = 0 .$$

Now the matrix of this system is the real part of the matrix $A(x_o)$ introduced in (3.7). Thus the vector $\{(k_i,g)\}$ in Z_n is in the

III-2

nullspace of the real part of $A(x_o)$. Next we claim that this vector is also in the nullspace of the imaginary part of $A(x_o)$. For, by assumption g is in $\mathcal{B}(\alpha|[\delta_1,\delta_2], \alpha-1, \lambda, \tilde{\mathcal{H}})$ hence it blows up near x_o at most as $(x-x_o)^{\alpha-1}$. Therefore the numerator of g vanishes at x_o, that is

$$\lambda_1 k_1(x_o)(k_1,g) + \ldots + \lambda_n k_n(x_o)(k_n,g) = 0.$$

Next, taking the inner product of this equation with each of the vectors $k_i(x_o)$ (in $\tilde{\mathcal{H}}$), yields,

$$\lambda_1(k_i(x_o),k_1(x_o))(k_1,g) + \ldots + \lambda_n(k_i(x_o),k_n(x_o))(k_n,g) = 0$$
$$i = 1,2,\ldots n.$$

Now in view of the definition of $A(x_o)$, that is in view of (3.7), the above equations show that the vector $\{(k_i,g)\}$ is in the nullspace of the imaginary part of $A(x_o)$. This completes the proof of the statement that the nullspace of $A(x_o)$ is non trivial, and the fact that $\det A(x_o) = 0$ is a necessary condition for the point x_o in $[\delta_1,\delta_2]$ to be a pointeigenvalue of M+K on $\mathcal{B}$.

<u>Remark</u>

Relation (III-2) shows that if x_o in $[\delta_1,\delta_2]$ is a pointeigenvalue of M+K on $\mathcal{L}_2(\lambda, \tilde{\mathcal{H}})$ then it is also a pointeigenvalue of M+K on $\mathcal{B}(\alpha|[\delta_1,\delta_2], \alpha-1,\lambda, \tilde{\mathcal{H}})$. Thus Condition $[\delta_1,\delta_2]$, which requires that the set of pointeigenvalues of M+K on $\mathcal{B}$ should be disjoint from the interval $[\delta_1,\delta_2]$, requires in particular, that this also holds for the operator M+K on $\mathcal{L}_2(\lambda, \tilde{\mathcal{H}})$. In general it requires more, however, there is a particular case in

which the two conditions are equivalent. This is the case of exponent $\alpha > 1/2$. For, for such values of α, we evidently have in inclusion:

$$\mathcal{B}(\alpha|[\delta_1,\delta_2],\alpha-1,\lambda,\mathfrak{H}) \subset \mathcal{L}_2(\lambda,\mathfrak{H}) .$$

Therefore, in this case the intersection of the set of point-eigenvalues of M+K on $\mathcal{L}_2(\lambda,\mathfrak{H})$ with the interval $[\delta_1,\delta_2]$ equals intersection of the set of point eigenvalues of M+K on $\mathcal{B}$ with the interval $[\delta_1,\delta_2]$.

Combining this fact with the present lemma we obtain <u>that the intersection of the set of pointeigenvalues of M+K on $\mathcal{B}$ or M+K on $\mathcal{L}_2$, with the interval $[\delta_1,\delta_2]$ equals the intersection of the set of zeros of det A(x) with the interval $[\delta_1,\delta_2]$.</u>

IV-1

APPENDIX IV

Lemma 1

Suppose that the accessory space is finite dimensional and that G is in $\mathfrak{G}(\alpha, 0, \tilde{\eta})$. Then the operator

$$(xy)^{-\beta}\, G(x,y)$$

is the sum of a gentle operator and a locally gentle operator of finite rank, for every $\beta < \min(\alpha, 1/2)$.

First consider the case in which

$$G(x,y) = 0 \quad \text{for } x = 0 \quad \text{or} \quad y = 0.$$

Then using that according to Nuskhelisvili, [[El,b]],

$$|x^{\alpha}[x+h)^{-} \quad -x \quad]| \leq h^{-\alpha}$$

a straightforward algebra shows the validity of the lemma.

Next consider the case in which G(x,y) has compact support, and let c be a continuously differentiable function with compact support such that $G(x,y) = c(x)c(y)G(x,y)$. Set

$$(xy)^{-\beta}G(x,y) = (xy)^{-\beta}c(x)c(y)[G(x,y) - G(0,y) - G(x,0) + G(0,0)]$$

$$\text{(IV.I)} \qquad + (xy)^{-\beta}c(x)c(y)[G(0,y) + G(x,0) - G(0,0)].$$

Then the first kernel vanishes for x=0 or y=0 and we see that it is in $\mathfrak{G}(\alpha-\beta, \beta, \tilde{\eta})$. On the other hand, by assumption, the accessory space is finite dimensional hence the second kernel defines an operator of finite rank and evidently it is locally gentle.

Finally let G be an arbitrary operator in $\mathfrak{G}(\alpha, \beta, \tilde{\eta})$, and set

$(xy)^{-\beta}G(x,y)=(xy)^{-\beta}c(x)c(y)G(x,y)+(xy)^{-\beta}(1-c(x)c(y))G(x,y).$

Then in view of (IV.1), the first term is the sum of a gentle operator in $\mathcal{G}(\alpha-\beta,\beta,\tilde{\mathfrak{N}})$ and a locally gentle operator of finite rank, and clearly the second term is gentle. This establishes the lemma.

<u>Lemma 2</u>

<u>Suppose that G is in $\mathcal{G}(\alpha,\beta,\tilde{\mathfrak{N}})$ and that the function f in $\mathcal{L}_2(\tilde{\mathfrak{N}})$ is gentle over $[\delta_1,\delta_2]$ with exponent α. Then the function ΓGf in $\mathcal{L}_2(\lambda,\tilde{\mathfrak{N}})$ is gentle over the same interval with the same exponent.</u>

According to [B5], ΓG is a bounded operator on $\mathcal{L}_2(\tilde{\mathfrak{N}})$, thus for every $\mathcal{L}_2$ function f, ΓGf is also in $\mathcal{L}_2$.

Following Friedrichs, we shall establish the gentleness of this function by introducing an auxiliary function of two variables,

(IV-2) $$F(u,x) = \int \frac{G(u,y)}{x-y} f(y)dy + i\pi\, G(u,x)f(x) .$$

Note that if G_u denotes the function obtained from G(u,y) by ''freezing the variable u'', then this equation can be written as

(IV-3) $$F(u,x) = H_+[G_u]f(x) \quad ;$$

remembering that $[G_u]$ is the multiplication operator associated with the function G_u. Next set

$$\Delta_h f(x) = (f(x+h) - f(x-h))(2h)^{-\alpha}$$

(IV-4) $$|f| = \sup_{\delta\leq x\leq\delta_2} f(x)$$

$$||f||_{\alpha,\delta} = \max \left\{|f|, \sup_{0\leq h\leq\rho} |\Delta_h f|\right\} .$$

Then using these notations, we have the following version of the Plemelj-Privalov theorem: To a given positive number ρ there is a number κ_ρ such that for every function

$$\|H_+ f\|_{\alpha,\rho} \leq \kappa_\rho \|f\|_{\alpha,\rho} \,. \tag{IV-5}$$

This theorem can be established in the same way as a slightly different version of the Plemelj-Privalov theorem was established by J. Schwartz in Lemma 3.2 of [   ].

Next consider the function $F(u,x)$ introduced in (IV-3). Let Δ_{h_1} be the operator in (IV-4) acting on the first variable of F and let Δ_{n_2} act on the second variable. Then with respect to the first variable, one can take Holder quotients under the integral sign in (IV-2), in other words, the operator Δ_{h_1} and H_+ commute; thus

$$|\Delta_{h_1} H_+[G_u]f| = |H_+[\Delta_{n_1} G_u]f| \,.$$

Now according to the definition of the gentleness norm in $\mathfrak{G}(\alpha,\beta,\mathfrak{H})$,

$$\|\Delta_{h_1} G_u\|_{\alpha,\rho} \leq \|G\|_\alpha \,.$$

Therefore (IV-5) yields

$$|H_+[\Delta_{h_1} G_u]f| \leq \kappa_\rho \|G\|_\alpha \|f\|_{\alpha,\rho} \,.$$

Since this relation holds for every value of h_1 we conclude from it that the function $F(u,x) = H_+[G_u]f(x)$ is Holder continuous in u over the interval $[\delta_1,\delta_2]$, and also over a slightly larger interval. On the other hand, the Holder continuity of this function in

x follows directly from (IV-5). Therefore, the function which we obtain from $F(u,x)$ by setting $u = x$ is also Holder continuous on the interval $[\delta_1,\delta_2]$, and also on a slightly larger interval. This establishes a lemma.

We shall also need a description of the function ΓGf in case f is in $\mathcal{B}(\alpha|[\delta_1,\delta_2],\gamma,\Omega)$ with $-1 < \gamma \leq 0$, which roughly speaking means that f is Holder continuous in $[\delta_1,\delta_2]$ and at some point of this interval it blows up like the fractional power γ, . In this case, instead of relation (IV-5), one needs an estimate of Muskhelishvili, describing H_+ acting on such functions, [E1,c]. Then an argument similar to the one used in establishing the present lemma, yields the following

Lemma 3

Suppose that G is in $\mathcal{G}(\alpha,\beta,\Omega)$ and that f is in $\mathcal{B}(\alpha|[\delta_1,\delta_2],\gamma,\Omega)$. Then the function ΓGf is in the same space $\mathcal{B}$.

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